



## **The compactness locus of a geometric functor and the formal construction of the Adams isomorphism**

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# THE COMPACTNESS LOCUS OF A GEOMETRIC FUNCTOR AND THE FORMAL CONSTRUCTION OF THE ADAMS ISOMORPHISM

BEREN SANDERS

**ABSTRACT.** We introduce the compactness locus of a geometric functor between rigidly-compactly generated tensor-triangulated categories, and describe it for several examples arising in equivariant homotopy theory and algebraic geometry. It is a subset of the tensor-triangular spectrum of the target category which, crudely speaking, measures the failure of the functor to satisfy Grothendieck-Neeman duality (or equivalently, to admit a left adjoint). We prove that any geometric functor — even one which does not admit a left adjoint — gives rise to a Wirthmüller isomorphism once one passes to a colocalization of the target category determined by the compactness locus. When applied to the inflation functor in equivariant stable homotopy theory, this produces the Adams isomorphism.

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## 1. INTRODUCTION

Motivated by a desire to clarify the relationship between Grothendieck duality in algebraic geometry and the Wirthmüller isomorphism in equivariant stable homotopy theory, P. Balmer, I. Dell’Ambrogio, and the present author recently made a general study [BDS16] of the existence and properties of adjoints to a geometric functor  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  between rigidly-compactly generated tensor-triangulated categories. (These definitions will be recalled in Section 2.) One highlight of that work was the realization that the Wirthmüller isomorphism is a general formal phenomenon, emerging (in our rigidly-compactly generated setting) as an isomorphism

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between the left and right adjoint of any tensor-triangulated functor — up to a twist by the relative dualizing object — whenever those adjoints exist. Moreover, we proved that the existence of those adjoints is equivalent to the original functor satisfying a form of Grothendieck duality.

More precisely, as explained in [BDS16], a geometric functor  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  between rigidly-compactly generated categories always admits a right adjoint  $f_*$  which itself admits a right adjoint  $f^!$ :

$$\begin{array}{ccc} & \mathcal{D} & \\ f^* \downarrow & & \uparrow f_* \downarrow f^! \\ & \mathcal{C} & \end{array}$$

Then, defining the *relative dualizing object* to be  $\omega_f := f^!(\mathbb{1}_{\mathcal{D}})$ , we proved:

**1.1. Theorem** ([BDS16, Thm. B]). *The following are equivalent:*

- (a) *The functor  $f^!$  admits a right adjoint.*
- (b) *There is a natural isomorphism  $f^! \simeq \omega_f \otimes f^*(-)$ .*
- (c) *The functor  $f_*$  preserves compact objects.*
- (d) *There is a natural isomorphism  $f^* \simeq \mathbf{hom}(\omega_f, f^!(-))$ .*
- (e) *The functor  $f^*$  admits a left adjoint  $f_!$ .*

Moreover, in this case, there is a canonical natural isomorphism

$$(1.2) \quad f_! \simeq f_*(- \otimes \omega_f).$$

We say that  $f^*$  satisfies *Grothendieck-Neeman duality* (or *GN-duality*, for short) when it satisfies the equivalent conditions (a)–(e) of Theorem 1.1. This terminology is motivated by the natural isomorphism in (b), exhibiting the double right adjoint  $f^!$  as a twisted version of the original functor  $f^*$ , as in classical algebro-geometric Grothendieck duality. On the other hand, when applied to the restriction functor  $f^* := \mathrm{res}_H^G : \mathrm{SH}(G) \rightarrow \mathrm{SH}(H)$  between equivariant stable homotopy categories, (1.2) specializes to the Wirthmüller isomorphism between induction and coinduction. In this way, Theorem 1.1 provides a purely formal, canonical construction of the classical Wirthmüller isomorphism in equivariant stable homotopy theory.

On the other hand, there is another important isomorphism in equivariant stable homotopy theory: the Adams isomorphism. Loosely speaking, it is an isomorphism (up to a twist) between the  $N$ -orbits and the  $N$ -fixed points of an  $N$ -free  $G$ -spectrum, and it lies at the heart of genuine equivariant stable homotopy theory, appearing for example in the tom Dieck splitting of equivariant homotopy groups. More precisely, given a closed normal subgroup  $N \trianglelefteq G$  of a compact Lie group  $G$ , it is a natural isomorphism of  $G/N$ -spectra

$$(1.3) \quad (i^* X \wedge E\mathcal{F}(N)_+)/N \cong \lambda^N(X \wedge S^{-\mathrm{Ad}(N;G)})$$

defined for any  $N$ -free  $G$ -spectrum  $X$ . (For the uninitiated, this notation will be explained in Section 3.) At naive first glance, (1.3) looks like it might be some kind of twisted isomorphism between a left and right adjoint, and it is then very natural to attempt to give it a formal treatment, just like Theorem 1.1 gave a formal treatment of the classical Wirthmüller isomorphism. Indeed, J.P. May raises the problem of giving a formal analysis of the Adams isomorphism in [May03].

Now, the functor  $\lambda^N : \mathrm{SH}(G) \rightarrow \mathrm{SH}(G/N)$  appearing on the right-hand side of (1.3) is the categorical fixed point functor, which is right adjoint to the inflation

functor  $\mathrm{infl}_{G/N}^G : \mathrm{SH}(G/N) \rightarrow \mathrm{SH}(G)$ . However, the tom Dieck splitting theorem implies that  $\lambda^N$  (with or without the twist by  $S^{-\mathrm{Ad}(N;G)}$ ) does not preserve compact objects, except in the trivial case when  $N = 1$  (see Proposition 3.2 below). It then follows from condition (c) of Theorem 1.1 that the inflation functor  $\mathrm{infl}_{G/N}^G$  does not have a left adjoint, and moreover, that the right-hand side of the Adams isomorphism (1.3) cannot be left adjoint to *any* functor between compactly generated categories which admits a right adjoint. Thus, naive attempts to understand (1.3) as a twisted isomorphism between a left and right adjoint do not succeed.

Nevertheless, we will show that the Adams isomorphism can be given a purely formal, conceptual construction, and that it can in fact be realized as a Wirthmüller isomorphism, properly understood. To this end, we continue the study initiated in [BDS16], now with a focus on functors  $f^*$  — such as  $\mathrm{infl}_{G/N}^G$  — which do not satisfy GN-duality (i.e. do not have a left adjoint). We can prove that every geometric functor unconditionally gives rise to a Wirthmüller isomorphism (and satisfies GN-duality) once one passes to a canonically determined finite colocalization of the target category:

**1.4. Theorem.** *Let  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  be a geometric functor between rigidly-compactly generated tensor-triangulated categories. Consider the thick  $\otimes$ -ideal*

$$\mathcal{A}_f := \{ x \in \mathcal{C}^c \mid f_*(x \otimes y) \text{ is compact for all } y \in \mathcal{C}^c \} \subseteq \mathcal{C}^c$$

*and let  $\Gamma^* : \mathcal{C} \rightarrow \mathcal{B} := \mathrm{Loc}\langle \mathcal{A}_f \rangle$  denote the associated finite colocalization, i.e. the right adjoint of the inclusion functor  $\Gamma_! : \mathcal{B} \hookrightarrow \mathcal{C}$ . Then:*

- (a) *The composite  $\Gamma^* \circ f^! : \mathcal{D} \rightarrow \mathcal{B}$  has a right adjoint.*
- (b) *There is a canonical natural isomorphism  $\Gamma^* \circ f^! \cong \Gamma^*(\omega_f \otimes f^*(-))$ .*
- (c) *There is a canonical natural isomorphism  $\Gamma^* \circ f^* \cong \Gamma^*(\mathrm{hom}(\omega_f, f^!(-)))$ .*
- (d) *The composite  $\Gamma^* \circ f^* : \mathcal{D} \rightarrow \mathcal{B}$  has a left adjoint, which we will denote*

$$(f \circ \Gamma)_! : \mathcal{B} \rightarrow \mathcal{D}$$

*and there is a canonical natural isomorphism*

$$(1.5) \quad (f \circ \Gamma)_!(x) \cong f_*(\Gamma_!(x) \otimes \omega_f)$$

*for all  $x \in \mathcal{B}$ .*

This theorem will be proved in Section 2. In the case that  $f^*$  does satisfy GN-duality, then  $\mathcal{B} = \mathcal{C}$  and (1.5) recovers the original Wirthmüller isomorphism (1.2). We shall see in Section 3 that when applied to the inflation functor  $f^* := \mathrm{infl}_{G/N}^G$ , the associated “Wirthmüller” isomorphism (1.5) is nothing but the Adams isomorphism. In this way, Theorem 1.4 provides a purely formal, conceptual, and canonical construction of the Adams isomorphism. Moreover, we obtain a unification of the Adams and Wirthmüller isomorphisms in equivariant stable homotopy theory; they arise by applying the same formal construction to inflation and restriction, respectively.

Now let’s take a step back. According to Theorem 1.4, every geometric functor  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  between rigidly-compactly generated tensor-triangulated categories has a canonically associated colocalization and concomitant Wirthmüller isomorphism. From the perspective of tensor-triangular geometry [Bal10b], the thick tensor-ideal  $\mathcal{A}_f$  appearing in the theorem corresponds to a certain Thomason sub-set

$$\mathrm{Z}_f \subset \mathrm{Spc}(\mathcal{C}^c)$$

of the tensor-triangular spectrum of the subcategory of compact objects  $\mathcal{C}^c$ . This subset  $Z_f \subset \mathrm{Spc}(\mathcal{C}^c)$  is a new invariant of the functor  $f^*$  which, crudely speaking, measures the failure of  $f^*$  to satisfy GN-duality (i.e. have a left adjoint). We call it the *compactness locus* of  $f^*$  (Def. 4.7) and initiate its general study in Section 4. The rest of the paper is devoted to understanding the compactness locus geometrically and computing it in specific examples.

We begin our geometric study in Section 5, where we give a complete topological description of the compactness locus of a finite localization (Proposition 5.3) and discuss several examples.

In Section 6, we complete our discussion of the Adams isomorphism by completely describing the compactness locus of the inflation functor  $\mathrm{infl}_{G/N}^G$  for any finite group  $G$  and normal subgroup  $N \trianglelefteq G$  (Theorem 6.2). We will see that the compactness locus singles out a larger subcategory than the subcategory of  $N$ -free  $G$ -spectra — which can be interpreted as saying that the “natural domain” of the Adams isomorphism is actually larger than the category of  $N$ -free  $G$ -spectra (cf. Rem. 6.15). Two explicit examples for  $G = D_{10}$  the dihedral group of order 10 are depicted in Figures 2–3 on pages 30–31. We restrict ourselves to finite groups in this section because it is only for finite groups that we have a description of the spectrum  $\mathrm{Spc}(\mathrm{SH}(G)^c)$  of the  $G$ -equivariant stable homotopy category (see [BS16]).

In Section 7, we study examples arising in algebraic geometry. Indeed, given any morphism  $f : X \rightarrow Y$  of (quasi-compact and quasi-separated) schemes, we can consider the compactness locus of the associated pull-back functor  $f^* : \mathrm{D}_{\mathrm{Qcoh}}(Y) \rightarrow \mathrm{D}_{\mathrm{Qcoh}}(X)$ . It is a certain Thomason subset

$$Z_f \subset X \cong \mathrm{Spc}(\mathrm{D}_{\mathrm{Qcoh}}(X)^c)$$

of the domain of the morphism  $f$ , which we would like to understand scheme-theoretically. One of our main theorems (Theorem 7.18) states that for a proper morphism  $f : X \rightarrow Y$  of noetherian schemes, the categorically-defined compactness locus  $Z_f$  is precisely the largest specialization closed subset of  $X$  contained in the scheme-theoretic perfect locus of the morphism  $f$ .

Finally, in Section 8, we give some miscellaneous additional examples and discuss directions for future research. Given its generality, it is perhaps not surprising that there exist geometric functors  $f^*$  whose compactness locus is empty, i.e. functors  $f^*$  for which the colocalization of Theorem 1.4 is the zero category. This can be seen in simple algebro-geometric examples, but Example 8.1 provides a non-trivial example of this phenomenon in stable homotopy theory. Nevertheless, it is quite satisfying that the property of having a left adjoint can be refined by a topological invariant associated to the functor, and it remains an interesting zoological challenge to analyze the compactness locus of any geometric functor we now find in nature.

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## 2. THE WIRTHMÜLLER ISOMORPHISM OF A GEOMETRIC FUNCTOR

Let’s begin with some standard terminology.

**2.1. Terminology.** By a *tensor-triangulated category*, we mean a triangulated category with a compatible closed symmetric monoidal structure as in [HPS97, App. A].

Such a category is *rigidly-compactly generated* if it is compactly generated as a triangulated category and if the compact objects coincide with the rigid objects (a. k. a. the strongly dualizable objects). In particular, the unit object  $\mathbb{1}$  is rigid-compact. In the language of [HPS97], this is precisely the same thing as a “unital algebraic stable homotopy category”. We will sometimes drop the “tensor-triangulated” and just speak of rigidly-compactly generated categories.

2.2. *Examples.* Several examples of rigidly-compactly generated categories are discussed in [BDS16, Examples 2.9–2.13]. Briefly, examples include:

- (a) The derived category  $D_{\text{Qcoh}}(X)$  of complexes of  $\mathcal{O}_X$ -modules having quasi-coherent homology, for  $X$  a quasi-compact and quasi-separated scheme.
- (b) The stable homotopy category  $\text{SH}$ .
- (c) The genuine  $G$ -equivariant stable homotopy category  $\text{SH}(G)$  for  $G$  a compact Lie group.
- (d) The stable module category  $\text{Stab}(\mathbb{k}G)$  for  $\mathbb{k}$  a field and  $G$  a finite group (or, more generally,  $G$  a finite group scheme over  $\mathbb{k}$ ).
- (e) The stable  $\mathbb{A}^1$ -homotopy category  $\text{SH}^{\mathbb{A}^1}(\mathbb{k})$  for a field  $\mathbb{k}$  of characteristic zero.
- (f) The derived category of motives  $\text{DM}(\mathbb{k}; R)$  for a field  $\mathbb{k}$  whose exponential characteristic is invertible in the coefficient ring  $R$ . (See [Tot16] and the references therein.)
- (g) The derived category  $D(A)$  of a highly-structured commutative ring spectrum  $A$  (e. g. a commutative dg-algebra).
- (h) Any smashing localization of a rigidly-compactly generated category (see e. g. [HPS97, Sec. 3.3]).

2.3. *Terminology.* By a *tensor-triangulated functor*, we mean a triangulated functor which is a *strong* monoidal functor. For brevity, a coproduct-preserving tensor-triangulated functor between rigidly-compactly generated categories will be called a *geometric functor*. (This terminology is motivated by [HPS97, Def. 3.4.1].)

2.4. *Hypothesis.* Throughout this section  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  will denote a geometric functor between rigidly-compactly generated categories. As recalled in the Introduction, any such functor admits two adjoints on the right,  $f^* \dashv f_* \dashv f^!$ , and we define the *relative dualizing object* of  $f^*$  to be  $\omega_f := f^!(\mathbb{1}_{\mathcal{D}})$ . (See [BDS16] for more details.)

2.5. *Remark* (The spectrum). Associated to  $\mathcal{C}$  is the topological space  $\text{Spc}(\mathcal{C}^c)$ , consisting of the prime tensor-ideals of the subcategory of compact objects  $\mathcal{C}^c$  (see [Bal05]). Every compact object  $x \in \mathcal{C}^c$  has an associated closed subset  $\text{supp}(x) := \{ \mathcal{P} \subset \mathcal{C}^c \mid x \notin \mathcal{P} \} \subset \text{Spc}(\mathcal{C}^c)$ , and these sets form a basis of closed sets for the topology on  $\text{Spc}(\mathcal{C}^c)$ . By the abstract Thick Subcategory Classification Theorem (see [Bal05, Thm 4.10] and [Bal05, Rem. 4.3]), the thick tensor-ideals of  $\mathcal{C}^c$  are in one-to-one correspondence with the *Thomason subsets* of  $\text{Spc}(\mathcal{C}^c)$  — i.e. the unions of closed sets, each of which has quasi-compact complement. The bijection sends a thick tensor-ideal  $\mathcal{J} \subset \mathcal{C}^c$  to the Thomason subset  $\bigcup_{x \in \mathcal{J}} \text{supp}(x)$ , while a Thomason subset  $Y \subset \text{Spc}(\mathcal{C}^c)$  is sent to the thick tensor-ideal  $\mathcal{C}_Y^c := \{ x \in \mathcal{C}^c \mid \text{supp}(x) \subset Y \}$ . It will also be useful to recall that the Thomason *closed* sets are precisely those of the form  $\text{supp}(x)$  for some compact  $x \in \mathcal{C}^c$  (cf. [Bal05, Prop 2.14]).

2.6. *Remark* (Finite localization). Let  $Y \subset \text{Spc}(\mathcal{C}^c)$  be a Thomason subset of the spectrum, with corresponding thick tensor-ideal  $\mathcal{C}_Y^c := \{ x \in \mathcal{C}^c \mid \text{supp}(x) \subset Y \}$ , and let  $V := \text{Spc}(\mathcal{C}^c) \setminus Y$  denote the complement. By the general theory of finite

localizations, there is an associated idempotent triangle

$$e_Y \rightarrow \mathbb{1} \rightarrow f_Y \rightarrow \Sigma e_Y$$

in  $\mathcal{C}$  (cf. [BF11, Sec. 2–3]), and we have identifications  $e_Y \otimes \mathcal{C} = \text{Loc}(e_Y) = \text{Loc}_{\otimes}(\mathcal{C}_Y^c) = \text{Loc}(\mathcal{C}_Y^c)$  and  $f_Y \otimes \mathcal{C} = \text{Loc}(\mathcal{C}_Y^c)^\perp$ . We denote the category of colocal objects (a.k.a. acyclic objects) by  $\mathcal{C}_Y := e_Y \otimes \mathcal{C}$  and the category of local objects by  $\mathcal{C}(V) := f_Y \otimes \mathcal{C}$ . The inclusion  $\mathcal{C}_Y \hookrightarrow \mathcal{C}$  of the colocal objects has a right adjoint  $e_Y \otimes - : \mathcal{C} \rightarrow \mathcal{C}_Y$ , called colocalization, and the inclusion  $\mathcal{C}(V) \hookrightarrow \mathcal{C}$  of the local objects has a left adjoint  $f_Y \otimes - : \mathcal{C} \rightarrow \mathcal{C}(V)$ , called localization. (See [BS16, Rem. 5.3] for a diagram of all the relevant functors.)

The category of local objects  $\mathcal{C}(V)$  inherits the structure of a tensor-triangulated category such that the localization functor  $\mathcal{C} \rightarrow \mathcal{C}(V)$  is a tensor-triangulated functor. It maps a set of compact-rigid generators of  $\mathcal{C}$  to a set of compact-rigid generators of  $\mathcal{C}(V)$ . In particular, the unit  $f_Y$  is compact in  $\mathcal{C}(V)$  and  $\mathcal{C}(V)$  is rigidly-compactly generated just as  $\mathcal{C}$  is. The localization functor preserves compact objects and hence induces a map on spectra  $\text{Spc}(\mathcal{C}(V)^c) \rightarrow \text{Spc}(\mathcal{C}^c)$  which identifies  $\text{Spc}(\mathcal{C}(V)^c) \cong V \subset \text{Spc}(\mathcal{C}^c)$ . This follows from the Thomason-Neeman localization theorem [Nee92, Thm. 2.1] which identifies  $\mathcal{C}(V)^c \cong (\mathcal{C}/\mathcal{C}_Y)^c \cong (\mathcal{C}^c/\mathcal{C}_Y^c)^\sharp$ .

On the other hand, the category of colocal objects  $\mathcal{C}_Y$  also inherits the structure of a tensor-triangulated category such that the colocalization functor  $e_Y \otimes - : \mathcal{C} \rightarrow \mathcal{C}_Y$  is a tensor-triangulated functor. By construction,  $\mathcal{C}_Y$  is compactly generated, but it is usually not rigidly-compactly generated since the unit object  $e_Y$  is usually not compact in  $\mathcal{C}_Y$ . Otherwise, since the inclusion  $\mathcal{C}_Y \hookrightarrow \mathcal{C}$  preserves compact objects, it would follow that  $e_Y$  is compact in  $\mathcal{C}$ , and this is only possible in the very special situation that  $Y$  is an open and closed subset of  $\text{Spc}(\mathcal{C}^c)$  (see the proof of Prop. 5.1 below). In the language of [HPS97], the category of colocal objects  $\mathcal{C}_Y$  is an algebraic stable homotopy category, but it is not a *unital* algebraic stable homotopy category (in general).

**2.7. Remark.** The notation  $\mathcal{C}_Y^c$  can now be interpreted in two ways: as  $(\mathcal{C}_Y)^c$  or as  $(\mathcal{C}^c)_Y$ . However, these coincide. More precisely, setting  $\mathcal{K} := \mathcal{C}^c$ , we have that  $(\mathcal{C}^c)_Y =: \mathcal{K}_Y = (\text{Loc}(\mathcal{K}_Y))^c = (\mathcal{C}_Y)^c$  where the middle equality is [Nee92, Lem. 2.2].

Our goal is to establish the following result:

**2.8. Theorem.** *Let  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  be a geometric functor between rigidly-compactly generated tensor-triangulated categories. Let  $\mathcal{J} \subseteq \mathcal{C}^c$  be a thick tensor-ideal of the category of compact objects  $\mathcal{C}^c$  such that  $f_*(\mathcal{J}) \subseteq \mathcal{D}^c$ . Let  $\mathcal{B} := \text{Loc}(\mathcal{J}) = \text{Loc}_{\otimes}(\mathcal{J})$  denote the localizing subcategory of  $\mathcal{C}$  generated by  $\mathcal{J}$  and let  $\Gamma^* : \mathcal{C} \rightarrow \mathcal{B}$  denote the associated finite colocalization (Rem. 2.6), i.e. the right adjoint of the inclusion  $\Gamma_! : \mathcal{B} \hookrightarrow \mathcal{C}$ . (Keep the diagram*

$$(2.9) \quad \begin{array}{ccc} & \mathcal{D} & \\ \begin{array}{c} \vdots \\ \uparrow f^* \\ \vdots \end{array} & \begin{array}{c} \uparrow \\ \downarrow f_* \end{array} & \begin{array}{c} \downarrow f^! \\ \vdots \end{array} \\ & \mathcal{C} & \\ \begin{array}{c} \uparrow \Gamma_! \\ \downarrow \Gamma^* \end{array} & \begin{array}{c} \uparrow \Gamma^* \\ \downarrow \Gamma_* \end{array} & \begin{array}{c} \vdots \\ \downarrow \Gamma_* \\ \vdots \end{array} \\ & \mathcal{B} & \end{array}$$

*in mind.) Then:*

- (a) The composite  $\Gamma^* \circ f^! : \mathcal{D} \rightarrow \mathcal{B}$  has a right adjoint.
- (b) There is a canonical natural isomorphism  $\Gamma^* \circ f^! \cong \Gamma^*(\omega_f \otimes f^*(-))$ .
- (c) There is a canonical natural isomorphism  $\Gamma^* \circ f^* \cong \Gamma^*(\text{hom}(\omega_f, f^!(-)))$ .
- (d) The composite  $\Gamma^* \circ f^* : \mathcal{D} \rightarrow \mathcal{B}$  has a left adjoint, which we will denote by

$$(f \circ \Gamma)_! : \mathcal{B} \rightarrow \mathcal{D}$$

and there is a canonical natural isomorphism

$$(2.10) \quad (f \circ \Gamma)_! \cong f_*(\Gamma_!(-) \otimes \omega_f)$$

which we call the Wirthmüller isomorphism associated to  $f^*$  and  $\mathcal{J}$ .

2.11. *Remark.* If  $f^*$  satisfies GN-duality (i.e. if it already admits a left adjoint) then we can just take  $\mathcal{J} := \mathcal{C}^c$ . In this case,  $\mathcal{B} = \mathcal{C}$  and the colocalization is just the identity. One can readily check (cf. Rem. 2.28 and [BDS16, (3.19)]) that the Wirthmüller isomorphism (2.10) reduces in this case to the original Wirthmüller isomorphism (1.2) of Theorem 1.1. The point of the new theorem is that it holds for functors which do not satisfy GN-duality.

2.12. *Remark.* The composite  $\Gamma^* \circ f^*$  also has a right adjoint  $f_* \circ \Gamma_*$ . However, the category  $\mathcal{B}$  is usually not rigidly-compactly generated and so we cannot apply the original Theorem 1.1 to the functor  $\Gamma^* \circ f^*$ . (The unit  $\mathbb{1}_{\mathcal{B}}$  is the left idempotent for the colocalization and is almost never compact, neither in  $\mathcal{B}$  nor in  $\mathcal{C}$ .) Indeed, since  $\Gamma^* \circ f^*$  does not preserve compact objects in general, its right adjoint  $f_* \circ \Gamma_*$  does not itself have a right adjoint. This demonstrates, in particular, that the Trichotomy Theorem of [BDS16, Cor. 1.13] can fail when the target category is not rigidly-compactly generated.

2.13. *Remark.* The functor appearing on the right-hand side of (2.10) uses the left adjoint  $\Gamma_!$  of  $\Gamma^*$  rather than its right adjoint  $\Gamma_*$ . Indeed, as mentioned in Remark 2.12, the right adjoint  $f_* \circ \Gamma_*$  (or a twisted version like  $f_*(\Gamma_*(-) \otimes \omega_f)$ ) is very unlikely to preserve coproducts and so have any chance of being a left adjoint. This is the precise issue that we mentioned in the Introduction (p. 3) about how the Adams isomorphism cannot naively be realized as an isomorphism of left and right adjoints.

2.14. *Remark.* Let  $\psi : f^*x \otimes \omega_f \rightarrow f^!(x)$  denote the map adjoint to

$$f_*(f^*x \otimes \omega_f) \underset{\text{proj}}{\simeq} x \otimes f_*\omega_f \xrightarrow{1 \otimes \epsilon} x \otimes \mathbb{1} \simeq x$$

and let  $\psi^{\text{ad}} : f^*x \rightarrow \text{hom}(\omega_f, f^!x)$  denote its adjoint. The precise statement of Theorem 2.8 (b)–(c) is that  $\Gamma^*(\psi)$  and  $\Gamma^*(\psi^{\text{ad}})$  are isomorphisms. Similarly, the Wirthmüller isomorphism (2.10) is precisely the isomorphism induced by  $\Gamma^*(\psi^{\text{ad}})$  by taking left adjoints. Thus, all three isomorphisms of the theorem are derived from the same source. Recognizing that the Wirthmüller isomorphism arises from  $\Gamma^*(\psi^{\text{ad}})$  clarifies the subtle point mentioned in Remark 2.13 that the right-hand side of (2.10) involves a mixture of left and right adjoints. While this might make formula (2.10) look strange when compared with the original (1.2), the two isomorphisms from which they spring,  $\Gamma^*(\psi^{\text{ad}})$  and  $\psi^{\text{ad}}$ , are perfectly harmonious.

2.15. *Remark.* Obtaining an “explicit” description of the relative dualizing object  $\omega_f$  is of course not formal. Indeed, the notion of “explicit” is subjective and naturally depends on the particular subject at hand — be it equivariant stable homotopy



theory, algebraic geometry, modular representation theory, and so on. Nevertheless, there is a useful formal procedure we can use to get a grasp on  $\omega_f$  by applying previously established work. Namely, suppose  $\Theta$  is an “explicit” object (i.e. an object we have explicitly defined using the methods of our particular subject) for which we have established that  $f_*(\Gamma_!(-) \otimes \Theta)$  is left adjoint to  $\Gamma^* \circ f^*$ . Then by taking right adjoints we have  $\Gamma^* \mathbf{hom}(\Theta, \omega_f) \cong \mathbb{1}_{\mathcal{B}}$  so that  $\Gamma^* \omega_f \cong \Gamma^* \Theta$  if, moreover, the object  $\Theta$  is invertible. If  $f^*$  satisfies GN-duality then of course  $\Gamma^*$  may be removed and we can conclude that  $\omega_f \cong \Theta$ , thereby giving an “explicit” description of the categorically defined  $\omega_f$ . As an example, this method shows that the relative dualizing object  $\omega_f$  of the restriction functor  $f^* := \text{res}_H^G : \text{SH}(G) \rightarrow \text{SH}(H)$  in equivariant stable homotopy theory is isomorphic to the representation sphere  $S^{L(H,G)}$  associated to the tangent representation of  $G/H$  at the identity coset (cf. the proof of [BDS16, Prop. 4.4]).

We devote the rest of this section to proving Theorem 2.8.

**2.16. Notation.** We write  $\Delta x := \mathbf{hom}(x, \mathbb{1})$  for the dual of a (not necessarily rigid) object  $x$ , and abbreviate  $\mathcal{C}(x, y) := \text{Hom}_{\mathcal{C}}(x, y)$ . In commutative diagrams, we will sometimes use the symbol  $\pi$  to indicate the use of the  $f^* \dashv f_*$  projection formula:  $f_*(f^* x \otimes y) \cong x \otimes f_* y$  (see [BDS16, (2.16)]).

**2.17. Lemma.** *The composite  $\Gamma^* \circ f^! : \mathcal{D} \rightarrow \mathcal{B}$  preserves coproducts, and hence has a right adjoint.*

*Proof.* For a set  $\{d_i\}_{i \in I}$  of objects in  $\mathcal{D}$ , let  $\varphi : \coprod_i \Gamma^* f^! d_i \rightarrow \Gamma^* f^! \coprod_i d_i$  denote the canonical map. By Yoneda, it suffices to prove that the natural map

$$(2.18) \quad \mathcal{B}(b, \coprod_i \Gamma^* f^! d_i) \xrightarrow{\varphi^\circ} \mathcal{B}(b, \Gamma^* f^! \coprod_i d_i)$$

is an isomorphism for all  $b \in \mathcal{B}$ . Since  $\mathcal{B}$  is compactly generated, it suffices to check this for  $b \in \mathcal{B}^c$  compact. Now, for  $b \in \mathcal{B}^c$  compact we have a chain of isomorphisms

$$\begin{aligned} \mathcal{B}(b, \coprod_i \Gamma^* f^! d_i) &\cong \oplus_i \mathcal{B}(b, \Gamma^* f^! d_i) && b \text{ is compact} \\ &\cong \oplus_i \mathcal{C}(\Gamma_! b, f^! d_i) && \text{adjunction } \Gamma_! \dashv \Gamma^* \\ &\cong \oplus_i \mathcal{D}(f_* \Gamma_! b, d_i) && \text{adjunction } f_* \dashv f^! \\ &\cong \mathcal{D}(f_* \Gamma_! b, \coprod_i d_i) && f_* \Gamma_! b \text{ is compact} \\ &\cong \mathcal{C}(\Gamma_! b, f^! \coprod_i d_i) && \text{adjunction } f_* \dashv f^! \\ &\cong \mathcal{B}(b, \Gamma^* f^! \coprod_i d_i) && \text{adjunction } \Gamma_! \dashv \Gamma^*. \end{aligned}$$

Here we have used that  $\Gamma_! b \in \text{Loc}(\mathcal{J})^c = \mathcal{J}$  (Rem. 2.7) and hence  $f_*(\Gamma_! b)$  is compact in  $\mathcal{D}$ . Then we just need to check that the composite isomorphism coincides with (2.18). This is a routine diagram chase. The existence of the right adjoint follows from Neeman’s Brown Representability Theorem for compactly generated categories (see e.g. [BDS16, Cor. 2.3]).  $\square$

**2.19. Proposition.** *The natural transformation*

$$\Gamma^*(\psi) : \Gamma^*(f^* x \otimes \omega_f) \rightarrow \Gamma^* f^! x$$

(cf. Rem. 2.14) *is a natural isomorphism for all  $x \in \mathcal{D}$ .*

$$(2.20) \quad \mathcal{B}(b, \Gamma^*(f^*x \otimes \omega_f)) \xrightarrow{\Gamma^*(\psi) \circ -} \mathcal{B}(b, \Gamma^*f^!x)$$
$$\begin{aligned}
\mathcal{B}(b, \Gamma^*(f^*x \otimes \omega_f)) &\cong \mathcal{C}(\Gamma_! b, f^*x \otimes \omega_f) \\
&\cong \mathcal{C}(\Gamma_! b \otimes \Delta f^*x, \omega_f) && f^*x \in \mathcal{C}^c \text{ is rigid} \\
&\cong \mathcal{C}(\Gamma_! b \otimes f^* \Delta x, \omega_f) && x \in \mathcal{D}^c \text{ is rigid} \\
&\cong \mathcal{D}(f_*(\Gamma_! b \otimes f^* \Delta x), \mathbb{1}_{\mathcal{D}}) \\
(2.21) \quad &\cong \mathcal{D}(f_* \Gamma_! b \otimes \Delta x, \mathbb{1}_{\mathcal{D}}) && \text{projection formula} \\
&\cong \mathcal{D}(f_* \Gamma_! b, x) && x \in \mathcal{D}^c \text{ is rigid} \\
&\cong \mathcal{C}(\Gamma_! b, f^! x) \\
&\cong \mathcal{B}(b, \Gamma^* f^! x).
\end{aligned}$$
$$\Gamma^*(f^*x \otimes \omega_f) \xrightarrow{\Gamma^*\eta} \Gamma^*f^!f_*(f^*x \otimes \omega_f) \xrightarrow{\Gamma^*f^!\pi} \Gamma^*f^!(x \otimes f_*\omega_f) \xrightarrow{\Gamma^*f^!(1 \otimes \epsilon)} \Gamma^*f^!x.$$
[illegible]
$$\begin{array}{ccc} f_*(\omega_f \otimes f^*x) \otimes \Delta x & \xrightarrow[\simeq]{-\pi} & f_*(\omega_f \otimes f^*x \otimes f^*\Delta x) \\ \pi^{-1} \otimes 1 \downarrow \simeq & & \downarrow \simeq \\ f_*\omega_f \otimes x \otimes \Delta x & \xrightarrow[\simeq]{\pi} & f_*(\omega_f \otimes f^*(x \otimes \Delta x)). \end{array}$$

is useful, which can be readily checked using the definition of the projection formula ([BDS16, (2.16)]).  $\square$

**2.22. Lemma.** *For any  $b \in \mathcal{B}^c$ , we have natural isomorphisms*

$$(2.23) \quad \mathcal{D}(\Delta f_* \Delta \Gamma_! (b), -) \cong \mathcal{C}(\Gamma_! (b), f^*(-)) \cong \mathcal{B}(b, \Gamma^* f^*(-))$$

*of functors  $\mathcal{D} \rightarrow \mathbf{Ab}$ .*

*Proof.* If  $b \in \mathcal{B}^c$  then  $\Gamma_! (b) \in \mathcal{J}$  (Rem. 2.7) and hence  $\Delta \Gamma_! (b) \in \mathcal{J}$  (since  $\mathcal{J}$  is a thick tensor-ideal and  $\Delta x$  is a direct summand of  $\Delta x \otimes x \otimes \Delta x$  for any rigid  $x$ ). Hence,  $f_*(\Delta \Gamma_! (b))$  is compact-rigid in  $\mathcal{D}$ . So for any  $d \in \mathcal{D}$ , we have a chain

$$\begin{aligned} \mathcal{D}(\Delta f_* \Delta \Gamma_! b, d) &\cong \mathcal{D}(\mathbb{1}_{\mathcal{D}}, f_* \Delta \Gamma_! b \otimes d) && f_* \Delta \Gamma_! b \in \mathcal{D}^c \text{ is rigid} \\ &\cong \mathcal{D}(\mathbb{1}_{\mathcal{D}}, f_*(\Delta \Gamma_! b \otimes f^* d)) && \text{projection formula} \\ &\cong \mathcal{C}(f^* \mathbb{1}_{\mathcal{D}}, \Delta \Gamma_! b \otimes f^* d) \\ &\cong \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \Delta \Gamma_! b \otimes f^* d) \\ &\cong \mathcal{C}(\Gamma_! b, f^* d) && \Gamma_! b \in \mathcal{C}^c \text{ is rigid} \\ &\cong \mathcal{B}(b, \Gamma^* f^* d) \end{aligned}$$

of isomorphisms natural in  $d$ .  $\square$

**2.24. Proposition.** *The composite  $\Gamma^* \circ f^* : \mathcal{D} \rightarrow \mathcal{B}$  preserves products, and hence has a left adjoint.*

*Proof.* For any set of objects  $\{d_i\}_{i \in I}$  in  $\mathcal{D}$ , let  $\varphi : \Gamma^* f^*(\Pi_i d_i) \rightarrow \Pi_i \Gamma^* f^*(d_i)$  denote the canonical map. By Yoneda, it suffices to prove that the natural map

$$(2.25) \quad \mathcal{B}(b, \Gamma^* f^*(\Pi_i d_i)) \xrightarrow{\varphi^\circ -} \mathcal{B}(b, \Pi_i \Gamma^* f^* d_i)$$

is an isomorphism for all  $b \in \mathcal{B}$ . Since  $\mathcal{B}$  is compactly generated, it suffices to check this for  $b \in \mathcal{B}^c$ . Now, for  $b \in \mathcal{B}^c$  we have a chain of isomorphisms

$$\begin{aligned} \mathcal{B}(b, \Gamma^* f^* \Pi_i d_i) &\cong \mathcal{D}(\Delta f_* \Delta \Gamma_! b, \Pi_i d_i) && \text{Lemma 2.22} \\ &\cong \Pi_i \mathcal{D}(\Delta f_* \Delta \Gamma_! b, d_i) \\ &\cong \Pi_i \mathcal{B}(b, \Gamma^* f^* d_i) && \text{Lemma 2.22} \\ &\cong \mathcal{B}(b, \Pi_i \Gamma^* f^* d_i) \end{aligned}$$

which we claim coincides with (2.25). This is a routine diagram chase using the naturality in  $d$  of the isomorphisms of Lemma 2.22. The existence of the left adjoint then follows from Neeman's Brown Representability Theorem for compactly generated categories (see e.g. [BDS16, Cor. 2.3]).  $\square$

**2.26. Notation.** We will denote the left adjoint of  $\Gamma^* \circ f^* : \mathcal{D} \rightarrow \mathcal{B}$  by

$$(f \circ \Gamma)_! : \mathcal{B} \rightarrow \mathcal{D}.$$

**2.27. Remark.** Lemma 2.22 implies that there is a canonical isomorphism  $(f \circ \Gamma)_! b \cong \Delta f_* \Delta \Gamma_! b$  for all compact  $b \in \mathcal{B}^c$ . More explicitly, chasing the unit for the  $(f \circ \Gamma)_! \dashv \Gamma^* \circ f^*$  adjunction through (2.23) we obtain an isomorphism  $\Delta f_* \Delta \Gamma_! b \rightarrow (f \circ \Gamma)_! b$  whose inverse  $(f \circ \Gamma)_! b \rightarrow \Delta f_* \Delta \Gamma_! b$  is the map adjoint to the map  $b \rightarrow \Gamma^* f^* \Delta f_* \Delta \Gamma_! b$  corresponding under (2.23) to the identity of  $\Delta f_* \Delta \Gamma_! b$ .

**2.28. Remark.** Consider the natural transformation

$$\psi^{\text{ad}} : f^* \rightarrow \text{hom}(\omega_f, f^!(-))$$

defined in Remark 2.14. Applying  $\Gamma^*$  we get a natural transformation

$$\Gamma^*(\psi^{\text{ad}}) : \Gamma^* \circ f^* \rightarrow \Gamma^* \text{hom}(\omega_f, f^!(-))$$

of functors  $\mathcal{D} \rightarrow \mathcal{B}$ . Taking left adjoints it corresponds to a natural transformation

$$(2.29) \quad \varpi : f_*(\Gamma_!(-) \otimes \omega_f) \rightarrow (f \circ \Gamma)_!$$

of functors  $\mathcal{B} \rightarrow \mathcal{D}$ . A long but straight-forward unravelling of the definitions shows that it is given explicitly for any  $x \in \mathcal{B}$  as the following composite:

$$\begin{array}{ccccc} f_*(\Gamma_!x \otimes \omega_f) & \xrightarrow{f_*(\Gamma_!\eta \otimes 1)} & f_*(\Gamma_!\Gamma^*f^*(f \circ \Gamma)_!x \otimes \omega_f) & \xrightarrow{f_*(\epsilon \otimes 1)} & f_*(f^*(f \circ \Gamma)_!x \otimes \omega_f) \\ \downarrow \varpi & & & & \downarrow \simeq \pi \\ (f \circ \Gamma)_!x & \xleftarrow{\simeq} & (f \circ \Gamma)_!x \otimes \mathbb{1} & \xleftarrow{1 \otimes \epsilon} & (f \circ \Gamma)_!x \otimes f_*\omega_f. \end{array}$$

**2.30. Proposition.** *The natural transformation  $\varpi : f_*(\Gamma_!x \otimes \omega_f) \rightarrow (f \circ \Gamma)_!$  (Rem. 2.28) is a natural isomorphism for all  $x \in \mathcal{B}$ .*

*Proof.* Note that both functors preserve coproducts, being (composites of) left adjoints. Hence, since  $\mathcal{B}$  is compactly generated, it suffices to check that  $\varpi$  is a natural isomorphism for  $x \in \mathcal{B}^c$ . Then by Yoneda, it suffices to prove that

$$(2.31) \quad \mathcal{D}(d, f_*(\Gamma_!x \otimes f^!\mathbb{1})) \xrightarrow{\varpi \circ -} \mathcal{D}(d, (f \circ \Gamma)_!x)$$

is an isomorphism for all  $d \in \mathcal{D}$  and all  $x \in \mathcal{B}^c$ . Now, for  $x \in \mathcal{B}^c$ , we have a chain of isomorphisms

$$\begin{aligned} \mathcal{D}(d, f_*(\Gamma_!x \otimes f^!\mathbb{1})) &\cong \mathcal{C}(f^*d, \Gamma_!x \otimes f^!\mathbb{1}) \\ &\cong \mathcal{C}(f^*d \otimes \Delta \Gamma_!x, f^!\mathbb{1}) && \Gamma_!x \in \mathcal{C}^c \text{ is rigid} \\ &\cong \mathcal{D}(f_*(f^*d \otimes \Delta \Gamma_!x), \mathbb{1}) \\ (2.32) \quad &\cong \mathcal{D}(d \otimes f_*\Delta \Gamma_!x, \mathbb{1}) && \text{projection formula} \\ &\cong \mathcal{D}(d, \Delta f_*\Delta \Gamma_!x) && f_*\Delta \Gamma_!x \in \mathcal{D}^c \text{ is rigid} \\ &\cong \mathcal{D}(d, (f \circ \Gamma)_!x) && \text{Remark 2.27} \end{aligned}$$

which we claim coincides with (2.31). Verifying this claim is quite involved, so for notational brevity, let us write  $\omega := \omega_f = f^!\mathbb{1}$  for the dualizing object,  $g_! := (f \circ \Gamma)_!$  for the left adjoint, and set  $c := \Gamma_!x$ . Then let  $u \in \mathcal{D}(d, f_*(c \otimes \omega))$  and consider the following diagram

$$\begin{array}{ccccc} d & \xrightarrow{\text{coev} \otimes 1} & \Delta f_*\Delta c \otimes f_*\Delta c \otimes d & \xrightarrow{1 \otimes \pi} & \Delta f_*\Delta c \otimes f_*(\Delta c \otimes f^*d) \\ \downarrow u & & \downarrow 1 \otimes 1 \otimes u & & \downarrow 1 \otimes f_*(1 \otimes f^*u) \\ f_*(c \otimes \omega) & \xrightarrow{\text{coev} \otimes 1} & \Delta f_*\Delta c \otimes f_*\Delta c \otimes f_*(c \otimes \omega) & \xrightarrow{1 \otimes \pi} & \Delta f_*\Delta c \otimes f_*(\Delta c \otimes f^*f_*(c \otimes \omega)) \\ \downarrow f_*(\Gamma_!\eta_x \otimes 1) & & \downarrow 1 \otimes \text{lax} & & \downarrow 1 \otimes f_*(1 \otimes \epsilon) \\ f_*(\Gamma_!\Gamma^*f^*g_!x \otimes \omega) & & & & \Delta f_*\Delta c \otimes f_*(\Delta c \otimes c \otimes \omega) \\ \downarrow f_*(\epsilon \otimes 1) & & & & \downarrow 1 \otimes f_*(\text{ev} \otimes 1) \\ f_*(f^*g_!x \otimes \omega) & & & & \Delta f_*\Delta c \otimes f_*\omega \\ \downarrow \pi \simeq & & & & \downarrow 1 \otimes \epsilon \\ g_!x \otimes f_*\omega & \xrightarrow{\Psi \otimes 1} & \Delta f_*\Delta c \otimes f_*\omega & & \\ \downarrow 1 \otimes \epsilon & & \downarrow 1 \otimes \epsilon & & \\ g_!x & \xrightarrow{\Psi} & \Delta f_*\Delta c & & \end{array}$$

where  $\Psi$  denotes the isomorphism of Remark 2.27. Going along the top, right-hand side, and bottom we get the image of  $u$  under the chained isomorphism (2.32) while going along the left-hand side we get the image of  $u$  under (2.31) (cf. Rem. 2.28). The triangular region can be checked directly from the definition of the projection formula. What remains is to check the interior region ( $\dagger$ ). This is a nightmarish diagram chase, so we will guide the reader through it.

We begin by obtaining a description of the isomorphism  $\Psi$ . To this end, let  $\delta : x \rightarrow \Gamma^* f^* \Delta f_* \Delta c$  denote the morphism corresponding to the identity of  $\Delta f_* \Delta c$  under the isomorphism (2.23). Chasing the identity through the definition of (2.23) we find that  $\delta$  equals the following morphism

$$(2.33) \quad x \xrightarrow{\eta} \Gamma^* c \simeq \Gamma^*(\mathbb{1} \otimes c) \xrightarrow{\Gamma^*(\zeta \otimes 1)} \Gamma^*(f^* \Delta f_* \Delta c \otimes \Delta c \otimes c) \xrightarrow{1 \otimes \text{ev}} \Gamma^* f^* \Delta f_* \Delta c$$

where  $\zeta : \mathbb{1} \rightarrow f^* \Delta f_* \Delta c \otimes \Delta c$  is defined by

$$\zeta : \mathbb{1} \simeq f^* \mathbb{1} \xrightarrow{\text{coev}} f^*(\Delta f_* \Delta c \otimes f_* \Delta c) \simeq f^* f_*(f^* \Delta f_* \Delta c \otimes \Delta c) \xrightarrow{\epsilon} f^* \Delta f_* \Delta c \otimes \Delta c.$$

Let  $\alpha_a : \Gamma^* a \otimes x \rightarrow \Gamma^*(a \otimes c)$  denote the map

$$\alpha_a : \Gamma^* a \otimes x \xrightarrow{1 \otimes \eta_x} \Gamma^* a \otimes \Gamma^* \Gamma_! x = \Gamma^* a \otimes \Gamma^* c \simeq \Gamma^*(a \otimes c)$$

which is natural in  $a$ . Then by applying naturality of  $\alpha$  with respect to  $\zeta$  one readily sees that (2.33) coincides with the following morphism

$$x \simeq \Gamma^* \mathbb{1} \otimes x \xrightarrow{\Gamma^* \zeta \otimes 1} \Gamma^*(f^* \Delta f_* \Delta c \otimes \Delta c) \otimes x \xrightarrow{\alpha} \Gamma^*(f^* \Delta f_* \Delta c \otimes \Delta c \otimes c) \xrightarrow{1 \otimes \text{ev}} \Gamma^* f^* \Delta f_* \Delta c$$

where we have abbreviated  $x \simeq \mathbb{1} \otimes x \simeq \Gamma^* \mathbb{1} \otimes x$ . Now, by definition, the isomorphism  $\Psi : g_! x \xrightarrow{\sim} \Delta f_* \Delta c$  of Remark 2.27 is given by

$$g_! x \xrightarrow{g_!(\delta)} g_!(\Gamma^* f^* \Delta f_* \Delta c) \xrightarrow{\epsilon} \Delta f_* \Delta c.$$

Using our second description of  $\delta$  (and consequently of  $\Psi$ ) we check that the bottom-left edge of ( $\dagger$ ) coincides with

$$\begin{aligned} f_*(\Gamma_! x \otimes \omega) &\simeq f_*(\Gamma_!(\Gamma^* \mathbb{1} \otimes x) \otimes \omega) \xrightarrow{\zeta} f_*(\Gamma_!(\Gamma^*(f^* \Delta f_* \Delta c \otimes \Delta c) \otimes x) \otimes \omega) \\ &\quad \downarrow \alpha \\ & f_*(\Gamma_! \Gamma^*(f^* \Delta f_* \Delta c \otimes \Delta c \otimes c) \otimes \omega) \\ &\quad \downarrow 1 \otimes \text{ev} \\ & f_*(\Gamma_! \Gamma^*(f^* \Delta f_* \Delta c) \otimes \omega) \\ &\quad \downarrow \epsilon \\ & f_*(f^* \Delta f_* \Delta c \otimes \omega) \\ &\quad \simeq \downarrow \pi \\ & \Delta f_* \Delta c \otimes f_* \omega. \end{aligned}$$

We are now in a position to check the commutativity of ( $\dagger$ ). Define

$$\beta : \Gamma_!(\Gamma^* a \otimes b) \xrightarrow{\text{colax}} \Gamma_! \Gamma^* a \otimes \Gamma_! b \xrightarrow{\epsilon \otimes 1} a \otimes \Gamma_! b.$$

The commutativity of  $(\dagger)$  can now be outlined as follows

(2.34)

where for example the first square is

$$\begin{array}{ccc}
 \Delta f_* \Delta c \otimes f_* \Delta c \otimes f_*(c \otimes \omega) & \xrightarrow{\pi} & f_*(f^*(\Delta f_* \Delta c \otimes f_* \Delta c) \otimes c \otimes \omega) \\
 \uparrow \text{coev} \otimes 1 & & \uparrow \beta \\
 f_*(\Gamma_! x \otimes \omega) \simeq f_*(\Gamma_!(\Gamma^* f^* \mathbb{1} \otimes x) \otimes \omega) & \xrightarrow{\text{coev}} & f_*(\Gamma_!(\Gamma^* f^*(\Delta f_* \Delta c \otimes f_* \Delta c) \otimes x) \otimes \omega)
 \end{array}$$

and the second square is

$$\begin{array}{ccc}
 f_*(f^*(\Delta f_* \Delta c \otimes f_* \Delta c) \otimes c \otimes \omega) & \xrightarrow{\pi} & f_*(f^* f^*(f^* \Delta f_* \Delta c \otimes \Delta c) \otimes c \otimes \omega) \\
 \uparrow \beta & & \uparrow \beta \\
 f_*(\Gamma_!(\Gamma^* f^*(\Delta f_* \Delta c \otimes f_* \Delta c) \otimes x) \otimes \omega) & \xrightarrow{\pi} & f_*(\Gamma_!(\Gamma^* f^* f^*(f^* \Delta f_* \Delta c \otimes \Delta c) \otimes x) \otimes \omega).
 \end{array}$$

The remaining squares should then be clear by following the guide (2.34). The commutativity of the second, third, fifth, and sixth squares follows immediately from naturality, while the commutativity of the fourth square follows from the definitions of  $\alpha$  and  $\beta$ . The first square requires a diagram chase but is fairly routine. Finally, the curved portion can be checked using the definition of the projection formula. This completes the proof, modulo the details we have only sketched.  $\square$

*Proof of Theorem 2.8.* Part (b) has been proved in Proposition 2.19. It follows immediately (or directly from Lemma 2.17) that (a) holds. Part (d) has been proved in Proposition 2.24 and Proposition 2.30. Part (c) then follows (Rem. 2.28) by taking right adjoints of the isomorphism (2.10) in Part (d).  $\square$

### 3. THE ADAMS ISOMORPHISM AS A WIRTHMÜLLER ISOMORPHISM

The purpose of this section is to show that the Adams isomorphism in equivariant stable homotopy theory can be realized as a special case of the formal Wirthmüller isomorphism of Theorem 2.8. To this end, let  $\text{SH}(G)$  denote the genuine  $G$ -equivariant stable homotopy category for a compact Lie group  $G$ . It is a rigidly-compactly generated tensor-triangulated category which is generated by the orbits  $\Sigma^\infty G/H_+$  as  $H \leq G$  ranges over all closed subgroups. (All subgroups are closed unless otherwise noted.) Any homomorphism of compact Lie groups  $f : G \rightarrow G'$  induces a tensor-triangulated functor  $f^* : \text{SH}(G') \rightarrow \text{SH}(G)$ . For example, the inclusion  $f : H \hookrightarrow G$  of a subgroup induces the restriction

functor  $\text{res}_H^G := f^* : \text{SH}(G) \rightarrow \text{SH}(H)$ . This geometric functor satisfies GN-duality, and the resulting Wirthmüller isomorphism is the eponymous classical Wirthmüller isomorphism (cf. [May03] and [BDS16, Ex. 4.5]). On the other hand, the quotient  $f : G \rightarrow G/N$  by a normal subgroup  $N \trianglelefteq G$  induces the inflation functor  $\text{infl}_{G/N}^G := f^* : \text{SH}(G/N) \rightarrow \text{SH}(G)$ . This is again a geometric functor whose right adjoint  $\lambda^N := f_* : \text{SH}(G) \rightarrow \text{SH}(G/N)$  is the so-called categorical  $N$ -fixed point functor. By the usual formal story, it itself admits a right adjoint  $f^! : \text{SH}(G/N) \rightarrow \text{SH}(G)$  which, to the author's knowledge, has not yet been studied (cf. Rem. 8.6). On the other hand, as Prop. 3.2 below shows,  $\text{infl}_{G/N}^G$  does not satisfy GN-duality (i.e. does not have a left adjoint) except in the trivial case when  $N = 1$ . To prove this, we will need the following standard facts about group cohomology:

**3.1. Proposition.** *Let  $G$  be a compact Lie group. Then  $H^*(BG; k)$  is not concentrated in finitely many degrees in either of the following cases:*

- (a)  $G$  is finite and  $k = \mathbb{F}_p$  with  $p$  dividing the order of  $G$ ;
- (b)  $G$  is not finite and  $k = \mathbb{Q}$ .

*Proof.* These facts follow from the Evens-Venkov Theorem, graded Noether normalization (e.g. [HS06, Thm. 4.2.3]), and Quillen's theorem that the Krull dimension of the cohomology ring  $H^*(BG; k)$  is equal to the  $p$ -rank of  $G$ , where  $p = \text{char } k$ . (By definition, the 0-rank of  $G$  is the rank of a maximal torus in  $G$ .)  $\square$

**3.2. Proposition.** *Let  $1 \neq N \trianglelefteq G$  be a nontrivial normal subgroup of a compact Lie group  $G$ . Then the categorical fixed point functor  $\lambda^N : \text{SH}(G) \rightarrow \text{SH}(G/N)$  does not preserve compact objects.*

*Proof.* According to the tom Dieck splitting theorem (cf. [Lew00, Cor. 3.4 (a)]), the  $N$ -fixed points  $\lambda^N(\mathbb{1})$  of the unit contains

$$(3.3) \quad \Sigma_{G/N}^\infty \left( E\mathcal{F}(N; G)_+ \wedge_N S^{\text{Ad}(N; G)} \right)$$

as a direct summand, where  $E\mathcal{F}(N; G)$  denotes the universal  $N$ -free  $G$ -space, and  $\text{Ad}(N; G)$  denotes the adjoint representation that conjugation by  $G$  induces on the Lie algebra of  $N$ . Thus, if  $\lambda^N(\mathbb{1})$  is compact then so is (3.3), and since  $\text{res}_1^{G/N}$  preserves compacts it would follow that

$$\begin{aligned} \text{res}_1^{G/N}(\Sigma_{G/N}^\infty E\mathcal{F}(N; G)_+ \wedge_N S^{\text{Ad}(N; G)}) &\cong \Sigma^\infty \text{res}_1^{G/N}(E\mathcal{F}(N; G)_+ \wedge_N S^{\text{Ad}(N; G)}) \\ &\cong \Sigma^\infty(EN_+ \wedge_N S^{\text{Ad}(N)}) \end{aligned}$$

is compact in  $\text{SH}$ . If  $N$  is finite then  $S^{\text{Ad}(N)} = S^0$  and this is just  $\Sigma^\infty BN_+$ , which is not compact in  $\text{SH}$ . Otherwise the cohomology groups

$$H\mathbb{F}_p^*(\Sigma^\infty BN_+) \cong \tilde{H}^*(BN_+; \mathbb{F}_p) \cong H^*(BN; \mathbb{F}_p)$$

would be concentrated in only finitely many degrees, contradicting Prop. 3.1 for any prime  $p$  dividing the order of  $N$ . If  $N$  is not finite, we similarly claim that  $\Sigma^\infty(EN_+ \wedge_N S^{\text{Ad}(N)})$  cannot be compact in  $\text{SH}$ . Indeed, otherwise

$$H\mathbb{Q}^*(\Sigma^\infty EN_+ \wedge_N S^{\text{Ad}(N)}) \cong \tilde{H}^*(EN_+ \wedge_N S^{\text{Ad}(N)}; \mathbb{Q})$$

would be concentrated in finitely many degrees. So to complete the proof we just need to establish the following claim: If  $G$  is a nonfinite compact Lie group then

the reduced Borel cohomology  $\tilde{H}_G^*(S^{\text{Ad}(G)}; \mathbb{Q})$  of the representation sphere for the adjoint representation is not concentrated in finitely many degrees.

For clarity, let  $X$  be any based  $G$ -space and write  $X_G := EG \times_G X$  for the (unreduced) Borel construction. For any subgroup  $K \leq G$ , we can consider the Leray-Serre spectral sequence for the fibration  $G/K \rightarrow X_K \rightarrow X_G$ . If this spectral sequence collapses, then we get an isomorphism  $H_K^*(X) \cong H_G^*(X) \otimes H^*(G/K)$ . Moreover, it will also follow that  $\tilde{H}_K^*(X) \cong \tilde{H}_G^*(X) \otimes H^*(G/K)$  by naturality of the spectral sequence with respect to the projection map  $X \rightarrow \text{pt}$  (after we recognize that  $\tilde{H}_G^*(X) \cong \text{coker}(H_G^*(\text{pt}) \rightarrow H_G^*(X))$ ). Thus, if  $\tilde{H}_G^*(X)$  were concentrated in finitely many degrees then the same would be true of  $\tilde{H}_K^*(X)$ . We apply this three times to reduce to the maximal torus: first to the connected component of the identity  $G_0 \leq G$ , then to the normalizer  $N_{G_0}T \leq G_0$  of a maximal torus  $T$  in  $G_0$ , and finally to  $T \leq N_{G_0}T$ . In all three cases the spectral sequence collapses because the associated orbit space  $G/K$  is rationally acyclic. Indeed,  $G/G_0$  and  $N_{G_0}T/T$  are finite, while for the remaining case  $G_0/N_{G_0}T$  see [Hsi75, p. 35, Lem. 1.1]. We are thus reduced to looking at  $\tilde{H}_T^*(X)$  for  $T$  a maximal torus.

Now, if  $X$  is a  $G$ -space whose cohomology groups are finite-dimensional in each degree and which is equivariantly formal (in the sense of [GKM98]) then the Leray-Hirsch theorem for the fibration  $X \rightarrow X_G \rightarrow BG$  will imply that  $\tilde{H}_G^*(X) \cong H^*(BG) \otimes \tilde{H}^*(X)$ . But when  $G = T$  is a torus, it is a consequence of the Borel localization theorem that we have an inequality  $\sum_i \dim H^i(X^T) \leq \sum_i \dim H^i(X)$  for any compact  $T$ -manifold  $X$ , with equality precisely when  $X$  is equivariantly formal (see [Hsi75, p. 46, Cor. 2]). Representation spheres  $X = S^V$  have the special property that their fixed points are again spheres. It thus follows that *any* representation sphere is equivariantly formal for the torus. Thus,  $\tilde{H}_T^*(S^{\text{Ad}(G)}) \cong H^*(BT) \otimes \tilde{H}^*(S^{\text{Ad}(G)}) \cong H^{*-\dim(G)}(BT)$  is not concentrated in finitely many degrees, by Prop. 3.1, which completes the proof.  $\square$

3.4. *Remark.* Since  $\text{infl}_{G/N}^G$  does not satisfy GN-duality, it is an excellent candidate for Theorem 2.8. Let's first set the stage by recalling some additional notions from equivariant homotopy theory.

3.5. *Remark.* Let  $\mathcal{F}$  be a family of subgroups of  $G$ , closed under conjugation and taking subgroups. Then we can consider the thick subcategory

$$(3.6) \quad \text{thick}(\Sigma^\infty G/K_+ \mid K \in \mathcal{F}) \subset \text{SH}(G)^c$$

generated by those orbits having isotropy in  $\mathcal{F}$ . The fact that  $\mathcal{F}$  is closed under subconjugation implies that (3.6) is actually a thick tensor-ideal of  $\text{SH}(G)^c$ . Indeed, it suffices to check that  $\Sigma^\infty G/H_+ \wedge \Sigma^\infty G/K_+ \cong \Sigma^\infty (G/H \times G/K)_+$  is contained in (3.6) for  $K \in \mathcal{F}$  and  $H \leq G$  arbitrary. For  $G$  finite this follows immediately from the double-coset decomposition of the product  $G/H \times G/K$ . For  $G$  arbitrary, we invoke a theorem of Illman [Ill83] that smooth compact  $G$ -manifolds are finite  $G$ -CW-complexes. We conclude that  $G/H \times G/K$  is built out of finitely many cells  $G/L \times D^n$  with  $L \in \mathcal{F}$  since each such cell contributes a point with isotropy  $L$  (and the isotropy of a point in  $G/H \times G/K$  is the intersection of a conjugate of  $H$  with a conjugate of  $K$ ).

3.7. *Remark.* The idempotent triangle for the finite localization (Rem. 2.6) associated to the thick tensor-ideal (3.6) is nothing but the exact triangle obtained by applying  $\Sigma^\infty$  to the isotropy cofiber sequence  $E\mathcal{F}_+ \rightarrow S^0 \rightarrow \tilde{E}\mathcal{F}$ . A  $G$ -spectrum  $X$



is colocal, that is  $E\mathcal{F}_+ \wedge X \simeq X$ , if and only if  $\Phi^H X = 0$  for all  $H \notin \mathcal{F}$ , while a  $G$ -spectrum  $X$  is local, that is  $\widetilde{E}\mathcal{F} \wedge X \simeq X$ , if and only if  $\Phi^H X = 0$  for all  $H \in \mathcal{F}$ . (Here we have dropped the suspensions, as usual, for readability.)

**3.8. Example.** For  $N \trianglelefteq G$  a normal subgroup we can take the family  $\mathcal{F}(N) := \{K \leq G \mid K \cap N = 1\}$ . The colocal objects are the so-called  $N$ -free  $G$ -spectra, i.e. those  $G$ -spectra  $X$  such that  $\Phi^H(X) = 0$  if  $H \cap N \neq 1$ .

**3.9. Example.** For  $N \trianglelefteq G$  a normal subgroup we can take the family  $\mathcal{F}[\not\supseteq N] := \{K \leq G \mid K \not\supseteq N\}$ . The local objects are the so-called  $N$ -concentrated  $G$ -spectra, i.e. those  $G$ -spectra  $X$  such that  $\Phi^H(X) = 0$  unless  $H \supseteq N$ .

**3.10. Lemma.** *Let  $N \trianglelefteq G$  be a normal subgroup of a compact Lie group  $G$ . For any compact  $N$ -free  $G$ -spectrum  $X \in \text{thick}\langle \Sigma^\infty G/K_+ \mid K \cap N = 1 \rangle \subset \text{SH}(G)^c$ , the  $N$ -fixed point spectrum  $\lambda^N(X)$  is compact in  $\text{SH}(G/N)$ .*

*Proof.* The collection of  $X \in \text{SH}(G)^c$  such that  $\lambda^N(X) \in \text{SH}(G/N)^c$  is a thick subcategory, so it suffices to check that  $\lambda^N(\Sigma^\infty G/K_+) \in \text{SH}(G/N)^c$  whenever  $K \cap N = 1$ . For any based  $G$ -space  $Y$ , the tom Dieck splitting theorem (cf. [Lew00, Corollary 3.4 (a)]) provides an isomorphism

$$\lambda^N(\Sigma_G^\infty Y) \cong \bigvee_{(H) \subseteq N} \Sigma_{G/N}^\infty G/N \rtimes_{WH} (E\mathcal{F}(W_N H; W_G H)_+ \wedge_{W_N H} \Sigma^{\text{Ad}(W_N H)} Y^H)$$

in  $\text{SH}(G/N)$ . Here the wedge is indexed over the  $G$ -conjugacy classes of subgroups  $H \leq N$ . Note that  $W_N H \trianglelefteq W_G H$  and  $WH := W_G H/W_N H \hookrightarrow G/N$  embeds as a subgroup. Moreover,  $E\mathcal{F}(W_N H; W_G H)$  denotes, as usual, the universal  $W_N H$ -free  $W_G H$ -space and  $\text{Ad}(W_N H)$  denotes the adjoint representation that  $W_G H$  induces on the Lie algebra of its normal subgroup  $W_N H$ . However,  $(G/K_+)^H = *$  if  $H \not\leq_G K$ , so the only nonzero summands in the tom Dieck splitting for an orbit  $Y = G/K_+$  are for those conjugacy classes of subgroups  $H \leq K \cap N$ . Thus, if  $K \cap N = 1$ , then it collapses to

$$(3.11) \quad \lambda^N(\Sigma_G^\infty G/K_+) \cong \Sigma_{G/N}^\infty (E\mathcal{F}(N; G)_+ \wedge_N \Sigma^{\text{Ad}(N; G)} G/K_+).$$

Moreover, the space  $G/K_+$  is  $N$ -free under our assumption  $K \cap N = 1$ , so (3.11) is just

$$\lambda^N(\Sigma_G^\infty G/K_+) \cong \Sigma_{G/N}^\infty ((G/K_+ \wedge S^{\text{Ad}(N; G)})/N)$$

which is certainly compact in  $\text{SH}(G/N)$ . Indeed, (a) the  $G$ -space  $G/K_+ \wedge S^{\text{Ad}(N; G)}$  has the homotopy type of a finite  $G$ -CW-complex, so its suspension  $G$ -spectrum (indexed on any  $G$ -universe) is compact, (b) the space-level and spectrum-level orbit functors intertwine the suspension functors to  $G/N$ -spectra and to  $G$ -spectra indexed on an  $N$ -trivial  $G$ -universe, and (c) the spectrum-level orbit functor (defined for  $G$ -spectra indexed on an  $N$ -trivial  $G$ -universe) preserves compact objects (as it is an exact functor between compactly generated categories with a double right adjoint).  $\square$

**3.12. Example.** Let  $N \trianglelefteq G$  be a normal subgroup of a compact Lie group  $G$  and let  $f^* := \text{infl}_{G/N}^G : \text{SH}(G/N) \rightarrow \text{SH}(G)$ . By Lemma 3.10 and Remark 3.5, the subcategory

$$\mathcal{J} := \text{thick}\langle G/H_+ \mid H \cap N = 1 \rangle \subset \text{SH}(G)^c$$

is a thick tensor-ideal of compact  $G$ -spectra, such that  $\lambda^N(\mathcal{J}) \subset \text{SH}(G/N)^c$ , and we can apply Theorem 2.8. Then  $\Gamma^* \cong E\mathcal{F}(N)_+ \wedge - : \text{SH}(G) \rightarrow N\text{-free-}\text{SH}(G) :=$

$\text{Loc}\langle G/H_+ \mid H \cap N = 1 \rangle$  is the colocalization onto the subcategory of genuine  $N$ -free  $G$ -spectra. As we now prove, the associated Wirthmüller isomorphism (2.10) can be identified as a natural isomorphism

$$(i^* X \wedge E\mathcal{F}(N)_+)/N \cong i^*(X \wedge \omega_f)^N$$

defined for all  $N$ -free  $G$ -spectra  $X$ , where  $i : \mathcal{U}^N \rightarrow \mathcal{U}$  denotes the inclusion of the  $N$ -fixed points of a complete  $G$ -universe  $\mathcal{U}$ . This requires some explanation.

Recall from [LMS86] that associated to  $i : \mathcal{U}^N \rightarrow \mathcal{U}$  is the change of universe adjunction

$$i_* : \text{SH}_{\mathcal{U}^N}(G) \rightleftarrows \text{SH}_{\mathcal{U}}(G) : i^*$$

which unfortunately clashes with our own notational conventions, since here  $i_*$  is left adjoint to  $i^*$ . We continue to write  $\text{SH}(G) = \text{SH}_{\mathcal{U}}(G)$  for the stable homotopy category of genuine  $G$ -spectra, but also write  $\text{SH}_{N\text{-triv}}(G) := \text{SH}_{\mathcal{U}^N}(G)$  for the stable homotopy category of  $G$ -spectra indexed on the  $N$ -trivial  $G$ -universe  $\mathcal{U}^N$ . Regarding  $\mathcal{U}^N$  as a complete  $G/N$ -universe, we have the change of group functor

$$\epsilon^* : \text{SH}(G/N) \cong \text{SH}_{\mathcal{U}^N}(G/N) \rightarrow \text{SH}_{\mathcal{U}^N}(G) = \text{SH}_{N\text{-triv}}(G)$$

which admits adjoints on both sides

$$\begin{array}{ccc} & \text{SH}(G/N) & \\ (-)/N \uparrow & \downarrow \epsilon^* & \uparrow (-)^N \\ & \text{SH}_{N\text{-triv}}(G) & \end{array}$$

the  $N$ -orbits and the  $N$ -fixed points. By definition, the inflation functor is the composite  $\text{infl}_{G/N}^G := i_* \circ \epsilon^* : \text{SH}(G/N) \rightarrow \text{SH}(G)$ , while the categorical  $N$ -fixed point functor is the composite  $\lambda^N := (i^*(-))^N : \text{SH}(G) \rightarrow \text{SH}(G/N)$ .

Since  $i_* \circ \Sigma_{\mathcal{U}^N}^\infty \cong \Sigma_{\mathcal{U}}^\infty$  and  $i_*$  is tensor-functor, we see that if  $X \in \text{SH}_{N\text{-triv}}(G)$  is  $N$ -free then  $i_* X$  is also  $N$ -free. (In contrast,  $i^*$  does not preserve  $N$ -free spectra in general.) Thus  $i_* : \text{SH}_{N\text{-triv}}(G) \rightarrow \text{SH}(G)$  restricts to a functor on the two categories of  $N$ -free spectra:  $i_* : N\text{-free-}\text{SH}_{N\text{-triv}}(G) \rightarrow N\text{-free-}\text{SH}(G)$ . The crucial fact about  $N$ -free  $G$ -spectra is that this functor is an equivalence ([LMS86, Theorem 2.8]). A quasi-inverse is given by  $E\mathcal{F}(N)_+ \wedge i^*(-)$ . Then contemplate the following diagram:

$$(3.13) \quad \begin{array}{ccccc} & & \text{SH}(G/N) & & \\ & \nearrow -/N & \uparrow \dashv & \downarrow \epsilon^* & \searrow \text{infl}_{G/N}^G \\ & & \text{SH}_{N\text{-triv}}(G) & & \\ & \nearrow \dashv & \downarrow E\mathcal{F}(N)_+ \wedge - & \searrow i_* & \\ & & N\text{-free-}\text{SH}_{N\text{-triv}}(G) & & \text{SH}(G) \\ & \nearrow E\mathcal{F}(N)_+ \wedge - & \downarrow i_* & \searrow E\mathcal{F}(N)_+ \wedge - & \\ & & \text{SH}_{N\text{-triv}}(G) & & \\ & \nearrow i^* & \downarrow (\dagger) & \searrow & \\ & & \text{SH}(G) & & \\ & \nearrow & \downarrow & \searrow & \\ & & N\text{-free-}\text{SH}(G) & & \end{array}$$

The two regions on the right commute, while the parallel arrows in the middle are adjoints. Here  $(\dagger)$  is the adjoint equivalence we just mentioned. In summary, we find that  $(E\mathcal{F}(N)_+ \wedge i^*(-))/N : N\text{-free-SH}(G) \rightarrow \text{SH}(G/N)$  is left adjoint to  $E\mathcal{F}(N)_+ \wedge \text{infl}_{G/N}^G = \Gamma^* \circ f^*$ . Thus, the Wirthmüller isomorphism (2.10) takes the form

$$(3.14) \quad (i^*X \wedge E\mathcal{F}(N)_+)/N \cong i^*(X \wedge \omega_f)^N.$$

That is, it is an isomorphism between the  $N$ -fixed points and the  $N$ -orbits of an  $N$ -free  $G$ -spectrum, up to a twist by the relative dualizing object  $\omega_f$ .

3.15. *Remark.* This isomorphism (3.14) is the Adams isomorphism. As explained in Remark 2.15, it is by no means formal to give an “explicit” description of the relative dualizing object  $\omega_f$ . In our present case, however, we can apply what is classically known about the Adams isomorphism (e.g. [LMS86, Thm. II.7.1]) via the method of Remark 2.15 to conclude that

$$(3.16) \quad E\mathcal{F}(N)_+ \wedge \omega_f \cong E\mathcal{F}(N)_+ \wedge S^{-\text{Ad}(N;G)}$$

where  $S^{-\text{Ad}(N;G)}$  is the inverse in  $\text{SH}(G)$  of the representation sphere for the adjoint  $G$ -representation  $\text{Ad}(N;G)$ . It follows that  $X \wedge \omega_f \cong X \wedge S^{-\text{Ad}(N;G)}$  for any  $N$ -free  $G$ -spectrum  $X$ , so the isomorphism (3.14) provided by Theorem 2.8 is indeed the Adams isomorphism

$$(i^*X \wedge E\mathcal{F}(N)_+)/N \cong i^*(X \wedge S^{-\text{Ad}(N;G)})^N$$

between  $N$ -fixed points and  $N$ -orbits up to a twist by the adjoint representation. However, it does not follow from (3.16) that  $\omega_f \cong S^{-\text{Ad}(N;G)}$  (i.e. before applying the colocalization  $E\mathcal{F}(N)_+ \wedge -$ ). It remains an interesting challenge to obtain an explicit description of the relative dualizing object of inflation and, more generally, of the right adjoint  $f^!$  to categorical fixed points. (See Remark 8.6 for more on this issue.)

#### 4. THE COMPACTNESS LOCUS OF A GEOMETRIC FUNCTOR

The statement of Theorem 2.8 involved the choice of a certain thick tensor-ideal  $\mathcal{J} \subset \mathcal{C}^c$ . Let us clarify the dependence on this choice.

4.1. **Lemma.** *Let  $\mathcal{J}_1 \subset \mathcal{J}_2$  be two thick tensor-ideals of  $\mathcal{C}^c$  such that  $f_*(\mathcal{J}_2) \subset \mathcal{D}^c$ . Then the inclusion  $\mathcal{B}_1 := \text{Loc}(\mathcal{J}_1) \hookrightarrow \text{Loc}(\mathcal{J}_2) =: \mathcal{B}_2$  has a right adjoint, so that the colocalization  $\Gamma_1^* : \mathcal{C} \rightarrow \mathcal{B}_1$  factors through the colocalization  $\Gamma_2^* : \mathcal{C} \rightarrow \mathcal{B}_2$ . It follows that we have a natural isomorphism*

$$(4.2) \quad (f \circ \Gamma_1)_!(x) \cong (f \circ \Gamma_2)_!(x)$$

for  $x \in \mathcal{B}_1$ . Moreover, under this isomorphism, the Wirthmüller isomorphism associated to  $\mathcal{J}_1$  coincides with the Wirthmüller isomorphism associated to  $\mathcal{J}_2$  restricted to  $\mathcal{B}_1$ .

*Proof.* That the two colocalizations nest is standard (consider the two associated idempotent triangles) and we thus get the isomorphism (4.2). The real thing that

needs to be checked is that the two Wirthmüller isomorphisms coincide — i.e., in pedantic notation, that

$$\begin{array}{ccc} f_*((\Gamma_1)_!x \otimes \omega_f) & \xrightarrow{\cong} & (f \circ \Gamma_1)_!(x) \\ \parallel & & \downarrow \cong \\ f_*((\Gamma_2)_!(\gamma_!x) \otimes \omega_f) & \xrightarrow{\cong} & (f \circ \Gamma_2)_!(\gamma_!x) \end{array}$$

commutes, where  $\gamma_!$  denotes the inclusion  $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ . This is a straight-forward but lengthy diagram-chase utilizing the description of the Wirthmüller isomorphism in Remark 2.28 and how the various isomorphisms between adjoints are defined in terms of the units and counits.  $\square$

**4.3. Remark.** Since the Wirthmüller isomorphism obtained by applying Theorem 2.8 to the subcategory  $\mathcal{J}_1$  is completely contained in what we obtain by applying the theorem to the larger subcategory  $\mathcal{J}_2$ , we really are only interested in maximal such  $\mathcal{J}$ . Well, there is a unique maximal such subcategory:

**4.4. Definition.** Let  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  be a geometric functor between rigidly-compactly generated tensor-triangulated categories. The *subcategory of  $f$ -relatively compact objects* is the thick tensor-ideal of  $\mathcal{C}^c$  defined as follows:

$$\mathcal{A}_f := \{ x \in \mathcal{C}^c \mid f_*(x \otimes y) \in \mathcal{D}^c \text{ for all } y \in \mathcal{C}^c \}.$$

It is the largest thick tensor-ideal of  $\mathcal{C}^c$  sent under  $f_*$  to  $\mathcal{D}^c$ .

**4.5. Remark.** According to Lemma 4.1, we should really only apply Theorem 2.8 to this specific thick tensor-ideal  $\mathcal{J} := \mathcal{A}_f$  which is canonically determined by  $f^*$ . We have stated (and proved) the theorem in terms of a chosen subcategory  $\mathcal{J}$  which could be smaller than  $\mathcal{A}_f$  for added flexibility in situations where  $\mathcal{A}_f$  is not yet explicitly understood.

**4.6. Remark.** The thick tensor-ideal  $\mathcal{A}_f$  corresponds to a certain Thomason subset  $Z_f \subset \mathrm{Spc}(\mathcal{C}^c)$  which we now single out for special attention.

**4.7. Definition.** Let  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  be a geometric functor between rigidly-compactly generated tensor-triangulated categories. The *compactness locus* of the functor  $f^*$  is the Thomason subset

$$Z_f := \bigcup_{x \in \mathcal{A}_f} \mathrm{supp}(x) \subset \mathrm{Spc}(\mathcal{C}^c)$$

corresponding to the thick tensor-ideal  $\mathcal{A}_f \subset \mathcal{C}^c$  (Def. 4.4).

**4.8. Remark.** The compactness locus  $Z_f \subset \mathrm{Spc}(\mathcal{C}^c)$  is a new topological invariant of the functor  $f^*$  and the remainder of this paper is devoted to its study. It is the whole space  $\mathrm{Spc}(\mathcal{C}^c)$  precisely when  $f^*$  satisfies GN-duality (cf. Thm. 1.1), but we would like to obtain a more refined and geometric understanding of  $Z_f$  in general. Before studying specific examples, we end this section with an alternative characterization of the  $f$ -relatively compact objects:

**4.9. Lemma.** *Let  $f^* : \mathcal{D} \rightarrow \mathcal{C}$  be a geometric functor between rigidly-compactly generated tensor-triangulated categories. A compact object  $x \in \mathcal{C}^c$  is contained in the subcategory  $\mathcal{A}_f \subset \mathcal{C}^c$  (Def. 4.4) if and only if the functor  $\mathrm{hom}(x, f^!(-)) \cong \Delta x \otimes f^!(-)$  preserves coproducts.*

*Proof.* Indeed, we have

$$\begin{aligned}
x \in \mathcal{A}_f &\Leftrightarrow f_*(x \otimes y) \text{ is compact in } \mathcal{D} \text{ for all } y \in \mathcal{C}^c \\
&\Leftrightarrow \mathcal{D}(f_*(x \otimes y), -) \text{ preserves coproducts for all } y \in \mathcal{C}^c \\
&\Leftrightarrow \mathcal{C}(x \otimes y, f^!(-)) \text{ preserves coproducts for all } y \in \mathcal{C}^c \\
&\Leftrightarrow \mathcal{C}(y, \mathbf{hom}(x, f^!(-))) \text{ preserves coproducts for all } y \in \mathcal{C}^c \\
&\Leftrightarrow \mathbf{hom}(x, f^!(-)) \text{ preserves coproducts}
\end{aligned}$$

where the only claim which is not immediate is the last  $\Rightarrow$ . To prove this, by Yoneda, we need only check that post-composition

$$\mathcal{C}(y, \Pi_i \mathbf{hom}(x, f^!(z_i))) \xrightarrow{\varphi \circ -} \mathcal{C}(y, \mathbf{hom}(x, f^!(\Pi_i z_i)))$$

by the canonical map  $\varphi : \Pi_i \mathbf{hom}(x, f^!(z_i)) \rightarrow \mathbf{hom}(x, f^!(\Pi_i z_i))$  is an isomorphism for all  $y \in \mathcal{C}$ . Since  $\mathcal{C}$  is compactly generated, we need only check this for  $y \in \mathcal{C}^c$  compact. But one can directly check that we have a factorization

$$\begin{array}{ccc}
\mathcal{C}(y, \Pi_i \mathbf{hom}(x, f^!(z_i))) & \xrightarrow{\varphi \circ -} & \mathcal{C}(y, \mathbf{hom}(x, f^!(\Pi_i z_i))) \\
\uparrow & \nearrow & \\
\Pi_i \mathcal{C}(y, \mathbf{hom}(x, f^!(z_i))) & & 
\end{array}$$

where the diagonal and vertical maps are isomorphisms (for  $y \in \mathcal{C}^c$ ).  $\square$

## 5. THE COMPACTNESS LOCUS OF A FINITE LOCALIZATION

We begin our study of the compactness locus by considering it for smashing and finite localizations. Recall (e. g. from [HPS97, Sec. 3.3]) that a smashing localization of a rigidly-compactly generated category  $\mathcal{C}$  is rigidly-compactly generated and the localization functor  $\mathcal{C} \rightarrow \mathcal{C}_L$  is a geometric functor.

**5.1. Proposition.** *A smashing localization  $\mathcal{C} \rightarrow \mathcal{C}_L$  satisfies GN-duality if and only if it is the finite localization associated to an open and closed subset  $Y \subseteq \mathrm{Spc}(\mathcal{C}^c)$ .*

*Proof.* ( $\Rightarrow$ ) Let  $e \rightarrow \mathbb{1} \rightarrow f \rightarrow \Sigma e$  be the idempotent triangle for the smashing localization. The right adjoint of localization is just the inclusion  $\mathcal{C}_L \hookrightarrow \mathcal{C}$  of the local objects  $\mathcal{C}_L = f \otimes \mathcal{C}$ . So GN-duality implies that the right idempotent  $f$  is compact as an object of  $\mathcal{C}$ . But then so is  $e$  (since  $\mathbb{1}$  is compact), so that  $e \rightarrow \mathbb{1} \rightarrow f \rightarrow \Sigma e$  is an idempotent triangle of compact objects. It follows that we have a decomposition  $\mathrm{Spc}(\mathcal{C}^c) = \mathrm{supp}(\mathbb{1}) = \mathrm{supp}(e) \sqcup \mathrm{supp}(f)$  into disjoint closed sets. In particular,  $Y := \mathrm{supp}(e)$  is an open and closed subset of  $\mathrm{Spc}(\mathcal{C}^c)$ . Moreover, since  $e$  is compact,  $\mathrm{Loc}_{\otimes}(e) = \mathrm{Loc}(\mathcal{C}_Y^c)$ , so indeed  $\mathcal{C} \rightarrow \mathcal{C}_L$  is nothing but the finite localization associated to the open and closed set  $Y = \mathrm{supp}(e)$ . ( $\Leftarrow$ ) Let  $\mathcal{C} \rightarrow \mathcal{C}(V)$  denote the finite localization associated to an open and closed subset  $Y \subseteq \mathrm{Spc}(\mathcal{C}^c)$  (so  $V := \mathrm{Spc}(\mathcal{C}^c) \setminus Y$  as usual). Applying the generalized Carlson connectedness theorem [Bal07, Thm. 2.11] to the decomposition  $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{C}^c) = Y \sqcup V$ , we conclude that  $\mathbb{1} \simeq a \oplus b$  for some  $a, b \in \mathcal{C}^c$  such that  $\mathrm{supp}(a) = Y$  and  $\mathrm{supp}(b) = V$ . Now let  $e \rightarrow \mathbb{1} \rightarrow f \rightarrow \Sigma e$  denote the idempotent triangle of the localization. We claim that it splits. First note that  $\mathrm{supp}(a) = Y$  implies  $a \in \mathcal{C}_Y^c$  is acyclic so that  $f \otimes a = 0$  and hence  $f \cong f \otimes b$ . On the other hand,  $\mathrm{supp}(b) \cap Y = \emptyset$  implies that  $b \otimes z = 0$  for all  $z \in \mathcal{C}_Y^c$ . Hence  $b \otimes e = 0$  since  $e \in \mathrm{Loc}(\mathcal{C}_Y^c)$ . Thus,

$e \cong e \otimes a$ . Now,  $\mathrm{Hom}_{\mathcal{C}}(f, \Sigma e) \cong \mathrm{Hom}_{\mathcal{C}}(f \otimes b, \Sigma e \otimes a) \cong \mathrm{Hom}(f, \Sigma e \otimes a \otimes \Delta b) = 0$  since  $\mathrm{supp}(a \otimes \Delta b) = \mathrm{supp}(a) \cap \mathrm{supp}(\Delta b) = \mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$ . So the sequence splits, and hence  $f$  is compact (being a direct summand of  $\mathbb{1}$ ). Now let's prove that the localization satisfies GN-duality. Well, if  $x \in \mathcal{C}(V)^c$  then the Thomason-Neeman localization theorem (cf. [Nee01, Cor. 4.5.14, Rem. 4.5.15]) implies  $x \oplus \Sigma x = f \otimes d$  for some  $d \in \mathcal{C}^c$ , which is compact in  $\mathcal{C}$ . Thus any  $x \in \mathcal{C}(V)^c$  is compact in  $\mathcal{C}$ , meaning that the inclusion  $\mathcal{C}(V) \hookrightarrow \mathcal{C}$  preserves compact objects.  $\square$

**5.2. Remark.** For finite localizations we completely understand the spectrum of the compact part of the localized category, and we can give a precise topological description of the compactness locus, as follows:

**5.3. Proposition.** *Let  $Y \subseteq \mathrm{Spc}(\mathcal{C}^c)$  be a Thomason subset, with complement  $V := \mathrm{Spc}(\mathcal{C}^c) \setminus Y$ , and let  $f^* : \mathcal{C} \rightarrow \mathcal{C}(V)$  denote the associated finite localization. Under the identification*

$$\mathrm{Spc}(\mathcal{C}(V)^c) \cong V \subseteq \mathrm{Spc}(\mathcal{C}^c)$$

*the compactness locus of  $f^*$ ,  $Z_f \subseteq V \subseteq \mathrm{Spc}(\mathcal{C}^c)$ , coincides with the largest Thomason subset of  $\mathrm{Spc}(\mathcal{C}^c)$  which is contained in  $V$ . In other words,*

$$Z_f = \bigcup_{\substack{x \in \mathcal{C}^c: \\ \mathrm{supp}(x) \subseteq V}} \mathrm{supp}(x) = \bigcup_{\substack{Z \subseteq \mathrm{Spc}(\mathcal{C}^c) \\ \text{Thomason closed:} \\ Z \subseteq V}} Z.$$

*Proof.* Let  $e \rightarrow \mathbb{1} \rightarrow f \rightarrow \Sigma e$  denote the idempotent triangle for the finite localization. We hope the double use of the letter  $f$  will cause no confusion. Indeed,  $f^*(-) = f \otimes -$  and  $\mathrm{Ker}(f \otimes -) = \mathrm{Loc}_{\otimes}(e) = \mathrm{Loc}(\mathcal{C}_Y^c)$ . Then observe that if  $x \in \mathcal{C}^c$  then  $\mathrm{supp}(x) \cap Y = \emptyset$  iff  $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$  for all  $y \in \mathcal{C}_Y^c$  iff  $x \otimes \mathcal{C}_Y^c = 0$  iff  $x \otimes \mathrm{Loc}(\mathcal{C}_Y^c) = 0$  iff  $x \otimes \mathrm{Loc}_{\otimes}(e) = 0$  iff  $x \otimes e = 0$ . Thus,  $f \otimes x \simeq x$  iff  $\mathrm{supp}(x) \subseteq V$ . Now consider  $a \in \mathcal{C}(V)^c$ . By the Thomason-Neeman localization theorem (cf. [Nee01, Cor. 4.5.14, Rem. 4.5.15]),  $a \oplus \Sigma a \simeq f \otimes b$  for some  $b \in \mathcal{C}^c$ . If  $a \in \mathcal{A}_f$  then so too  $a \oplus \Sigma a \in \mathcal{A}_f$  and hence  $f \otimes b$  is compact in  $\mathcal{C}$ . Thus

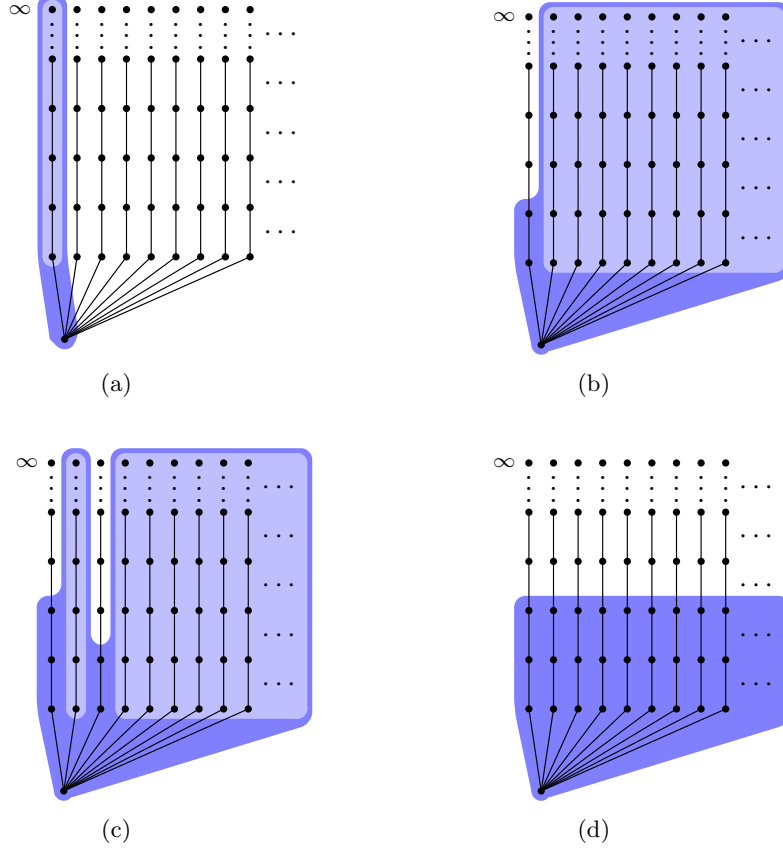
$$(5.4) \quad Z_f \subseteq \bigcup_{\substack{b \in \mathcal{C}^c: \\ f \otimes b \in \mathcal{C}^c}} \mathrm{supp}(f \otimes b) \subseteq \bigcup_{\substack{x \in \mathcal{C}^c: \\ f \otimes x \simeq x}} \mathrm{supp}(x)$$

where the second inequality follows just by taking  $x := f \otimes b$ . On the other hand, consider  $x \in \mathcal{C}^c$  such that  $f \otimes x \simeq x$ . For any  $y \in \mathcal{C}(V)^c$  we have  $y \oplus \Sigma y \simeq f \otimes c$  for some  $c \in \mathcal{C}^c$ , and then  $x \otimes (y \oplus \Sigma y) \simeq x \otimes f \otimes c \simeq x \otimes c$  is compact in  $\mathcal{C}$ . Hence its  $\oplus$ -summand  $x \otimes y$  is also compact in  $\mathcal{C}$ . This shows that if  $x \in \mathcal{C}^c$  satisfies  $f \otimes x \simeq x$  then  $x \in \mathcal{A}_f$ , showing that the inequalities in (5.4) are equalities.  $\square$

**5.5. Example.** The  $p$ -localization  $\mathrm{SH} \rightarrow \mathrm{SH}_{(p)}$  of the stable homotopy category is an example of a finite localization. Indeed, if  $\mathcal{C}_{p,n} \in \mathrm{Spc}(\mathrm{SH}^c)$  denotes the kernel in  $\mathrm{SH}^c$  of the  $(n-1)$ th Morava K-theory (after localization at  $p$ ), then  $p$ -localization is the finite localization associated to the Thomason subset  $Y = \bigcup_{q \neq p} \overline{\{\mathcal{C}_{q,2}\}}$ . (We refer the reader to [BS16] or [Bal10a, Sec. 9] for a description of the space  $\mathrm{Spc}(\mathrm{SH}^c)$ .) Then  $\mathrm{Spc}(\mathrm{SH}_{(p)}^c) \cong V \subseteq \mathrm{Spc}(\mathrm{SH}^c)$  identifies the spectrum of the  $p$ -local category with  $V = \{\mathcal{C}_{p,n} \mid 1 \leq n \leq \infty\}$ . One immediately sees from Proposition 5.3 that the compactness locus of  $\mathrm{SH} \rightarrow \mathrm{SH}_{(p)}$  is

$$Z_f = \{\mathcal{C}_n \mid 2 \leq n \leq \infty\} \subset \mathrm{Spc}(\mathrm{SH}_{(p)}^c).$$

That is, it is the whole of the  $p$ -local spectrum except for a single point — the generic point. (See diagram (a) below.) In fact, we can easily use Proposition 5.3, to describe the compactness locus of any finite localization of SH. For illustrative purposes, we have drawn some examples below. In each example, we have drawn the three subsets  $Z_f \subset V \subset X$ ; the light blue region is  $Z_f$  and the dark blue region is  $V$ . (Note that in example (d),  $Z_f$  is empty.)



5.6. *Example.* Besides  $\mathcal{C} = \text{SH}$ , we can also apply Proposition 5.3 to some other interesting examples, notably  $\mathcal{C} = \text{D}_{\text{Qcoh}}(X)$  for a quasi-compact and quasi-separated scheme  $X$ . Indeed, open immersions and the “inclusion of a stalk” are both examples of finite localizations on  $\text{D}_{\text{Qcoh}}(X)$  so Proposition 5.3 applies. For example, if  $f : U \hookrightarrow X$  is the inclusion of a quasi-compact open subset then the compactness locus  $Z_f \subset U \cong \text{Spc}(\text{D}_{\text{Qcoh}}(U)^c)$  of the derived pull-back  $f^* : \text{D}_{\text{Qcoh}}(X) \rightarrow \text{D}_{\text{Qcoh}}(U)$  is precisely the union of all Thomason closed subsets of  $X \cong \text{Spc}(\text{D}_{\text{Qcoh}}(X)^c)$  which are contained in  $U$ .

5.7. *Remark.* If  $X = \text{Spc}(\mathcal{C}^c)$  is noetherian, then  $X$  has finitely many connected components and the components are both open and closed. In this case, a subset  $Y \subseteq X$  is open and closed iff it is a (possibly empty) union of connected components. Note also that when  $X$  is noetherian we do not need to distinguish between closed subsets and Thomason closed subsets, so our understanding of Proposition 5.3

simplifies somewhat. In this case,  $Z_f$  is the union of all closed subsets of  $X$  which are contained in  $V$  — i.e. which do not intersect  $Y$  — or, thinking more geometrically, a point  $x \in X$  is contained in the compactness locus iff  $\overline{\{x\}} \subset V$ . In other words,  $Z_f$  and  $V$  have the same closed points, but  $Z_f$  really consists of those points  $x$  which are not contained in  $Y$ , and for which, moreover, the entire irreducible closed set  $\overline{\{x\}}$  does not intersect  $Y$ . This kind of interplay has to do with the fact that  $Z_f$  is closed under specialization while  $V$  is closed under generalization. In any case, the assumption that  $X$  is noetherian is sometimes overkill for these considerations. For example, although the space  $X = \mathrm{Spc}(\mathrm{SH}^c)$  in Example 5.5 is not noetherian, the compactness locus of any finite localization coincides with the union of all (not necessarily Thomason) closed subsets contained in  $V$ . In any case, let us end this section by giving an explicit example where these considerations do matter.

5.8. *Example.* Let  $\mathcal{C} = \mathrm{D}(\mathbb{F}_2^{\mathbb{N}})$  be the derived category of a countably infinite product of the field  $\mathbb{F}_2$ . The spectrum  $X := \mathrm{Spc}(\mathcal{C}^c) = \mathrm{Spec}(\mathbb{F}_2^{\mathbb{N}})$  is homeomorphic to the Stone-Ćech compactification of  $\mathbb{N}$  — that is, the Cantor space. Since the Cantor space is compact Hausdorff, we have that a subset is (quasi-)compact iff it is closed. Thus, the Thomason closed subsets are precisely the clopen subsets. Moreover, since the clopen subsets form a basis for the topology on the Cantor space, we see that the Thomason subsets are precisely the open subsets. Then take any point  $x \in X$ . It is closed (as  $X$  is totally disconnected), hence the complement  $Y := X \setminus \{x\}$  is open, and hence Thomason. Consider the associated finite localization of  $\mathrm{D}(\mathbb{F}_2^{\mathbb{N}})$  associated to  $Y$ . According to Proposition 5.3,  $Z_f$  is the union of all Thomason subsets of  $X$  which are contained in  $V = \{x\}$ . But the point  $x$  is not open (every open set is uncountable) so  $Z_f = \emptyset$ . On the other hand, if we took the union of all (not necessarily Thomason) closed subsets of  $X$  which are contained in  $V$ , then we would obtain  $V$  itself.

## 6. THE COMPACTNESS LOCUS OF INFLATION

We have seen that by taking  $\mathcal{J}$  to be the subcategory of finite  $N$ -free  $G$ -spectra, then Theorem 2.8 applies and provides the Adams isomorphism (Example 3.12). But really one should apply the theorem to the canonically determined subcategory associated to the compactness locus of the functor (Remark 4.5). Does the compactness locus of inflation single out precisely the  $N$ -free  $G$ -spectra, or does it single out a larger collection of  $G$ -spectra? In this section, we will answer this question, by figuring out precisely what the compactness locus of inflation is. This is possible, because we know what the spectrum of  $\mathrm{SH}(G)^c$  looks like, by [BS16], up to an unresolved indeterminacy in the topology, related to the chromatic shifting behaviour of the Tate construction. Even though this indeterminacy in the topology is presently unresolved, we know enough about the topology to completely describe the compactness locus of inflation (Theorem 6.2 below). All of this will be for finite groups  $G$  as it is only in this case that we have a description of the spectrum  $\mathrm{Spc}(\mathrm{SH}(G)^c)$ .

We require that the reader has some familiarity with [BS16]. Recall that for a finite group  $G$ , the topological space  $\mathrm{Spc}(\mathrm{SH}(G)^c)$  consists of points

$$\mathcal{P}(H, p, n) := \{ X \in \mathrm{SH}(G)^c \mid \Phi^H(X) \in \mathcal{C}_{p,n} \text{ in } \mathrm{SH}^c \}$$



for each conjugacy class of subgroups  $H \leq G$ , prime number  $p$ , and “chromatic” integer  $1 \leq n \leq \infty$ , where  $\Phi^H : \mathrm{SH}(G)^c \rightarrow \mathrm{SH}^c$  denotes the geometric  $H$ -fixed point functor. These points are all distinct except when  $n = 1$  where we have  $\mathcal{P}(H, p, 1) = \mathcal{P}(H, q, 1)$  for all primes  $p, q$ ; consequently, we will just write  $\mathcal{P}(H, 1)$  for this point. We also sometimes write  $\mathcal{P}_G(H, p, n)$  to indicate the ambient group  $G$ . Finally, recall that  $\mathrm{O}^p(G)$  denotes the smallest normal subgroup of  $G$  with index a power of  $p$ .

6.1. *Notation.* For a normal subgroup  $N \trianglelefteq G$ , we write

$$\mathcal{Z}_{N,G} \subset \mathrm{Spc}(\mathrm{SH}(G)^c)$$

for the compactness locus of the inflation functor  $\mathrm{infl}_{G/N}^G : \mathrm{SH}(G/N) \rightarrow \mathrm{SH}(G)$  and write  $\mathcal{A}_{N,G} \subset \mathrm{SH}(G)^c$  for the corresponding thick tensor-ideal.

The ultimate goal of this section is to prove the following theorem:

6.2. **Theorem.** *Let  $G$  be a finite group and let  $N \trianglelefteq G$  be a normal subgroup. The compactness locus  $\mathcal{Z}_{N,G} \subset \mathrm{Spc}(\mathrm{SH}(G)^c)$  of  $\mathrm{infl}_{G/N}^G : \mathrm{SH}(G/N) \rightarrow \mathrm{SH}(G)$  may be described as follows. For any  $H \leq G$ , we have the following:*

- (a) *If  $N \cap H \not\subseteq \mathrm{O}^p(H)$ , then  $\mathcal{P}_G(H, p, n) \notin \mathcal{Z}_{N,G}$  for all  $2 \leq n \leq \infty$ .*
- (b) *If  $N \cap H \subseteq \mathrm{O}^p(H)$ , then  $\mathcal{P}_G(H, p, n) \in \mathcal{Z}_{N,G}$  for all  $2 \leq n \leq \infty$ .*
- (c)  *$\mathcal{P}_G(H, 1) \in \mathcal{Z}_{N,G}$  if and only if  $N \cap H \subseteq \mathrm{O}^p(H)$  for all primes  $p$ .*

This will be proved on page 28, after we have developed the necessary lemmas. Some explicit examples will be drawn in Example 6.13 and Example 6.14.

6.3. **Lemma.** *Let  $K \leq H \leq G$  be finite and let  $x \in \mathrm{SH}(H)^c$ . If  $\mathcal{P}_H(K, p, n) \in \mathrm{supp}(x)$  then  $\mathcal{P}_G(K, p, n) \in \mathrm{supp}(\mathrm{ind}_H^G(x))$ .*

*Proof.* The hypothesis means that  $x \notin \mathcal{P}_H(K, p, n)$ , i.e. that  $\Phi^K(\mathrm{res}_K^H(x)) \notin \mathcal{C}_{p,n}$ . On the other hand, the conclusion means that  $\mathrm{ind}_H^G(x) \notin \mathcal{P}_G(K, p, n)$ , i.e. that  $\Phi^K(\mathrm{res}_K^G \mathrm{ind}_H^G(x)) \notin \mathcal{C}_{p,n}$ . But  $\Phi^K(\mathrm{res}_K^G \mathrm{ind}_H^G(x)) \cong \Phi^K(\mathrm{res}_K^H \mathrm{res}_H^G \mathrm{ind}_H^G(x))$  and  $x$  is a direct summand of  $\mathrm{res}_H^G \mathrm{ind}_H^G(x)$  (see e.g. [BDS15, Lem. 3.3]). Thus the hypothesis implies the conclusion.  $\square$

6.4. **Proposition.** *Let  $G$  be a finite group and let  $N \trianglelefteq G$  and  $H \leq G$ . Then  $\mathcal{P}_G(H, p, n) \in \mathcal{Z}_{N,G}$  if and only if  $\mathcal{P}_H(H, p, n) \in \mathcal{Z}_{H \cap N, H}$ .*

*Proof.* First we prove the  $(\Rightarrow)$  direction. Let  $x \in \mathcal{A}_{N,G}$  with  $\mathcal{P}_G(H, p, n) \in \mathrm{supp}(x)$ . Then  $\mathcal{P}_H(H, p, n) \in \mathrm{supp}(\mathrm{res}_H^G(x))$  by [BS16, Cor. 4.4], so it suffices to prove that  $\mathrm{res}_H^G(x) \in \mathcal{A}_{H \cap N, H}$ . For any  $y \in \mathrm{SH}(H)^c$ , we have that

$$\lambda^{H \cap N, H}(\mathrm{res}_H^G(x) \wedge y) \in \mathrm{SH}(H/H \cap N) \cong \mathrm{SH}(HN/N)$$

is a direct summand of

$$\begin{aligned} \mathrm{res}_{HN/N}^{G/N} \mathrm{coind}_{HN/N}^{G/N} \lambda^{H \cap N, H}(\mathrm{res}_H^G(x) \wedge y) &\cong \mathrm{res}_{HN/N}^{G/N} \lambda^{N, G} \mathrm{coind}_H^G(\mathrm{res}_H^G(x) \wedge y) \\ &\cong \mathrm{res}_{HN/N}^{G/N} \lambda^{N, G}(x \wedge \mathrm{coind}_H^G(y)) \end{aligned}$$

which is compact since restriction and coinduction both preserve compact objects and  $x \in \mathcal{A}_{N,G}$  by assumption. Hence  $\mathrm{res}_H^G(x) \in \mathcal{A}_{H \cap N, H}$ . Now we prove the  $(\Leftarrow)$  direction. Let  $x \in \mathcal{A}_{H \cap N, H}$  with  $\mathcal{P}_H(H, p, n) \in \mathrm{supp}(x)$ . We claim that  $\mathrm{ind}_H^G(x)$  is

contained in  $\mathcal{A}_{N,G}$ . The conclusion will then follow since Lemma 6.3 implies that  $\mathcal{P}_G(H, p, n) \in \text{supp}(\text{ind}_H^G(x))$ . Now,  $x \in \mathcal{A}_{H \cap N, H}$  implies that

$$\lambda^{H \cap N, H}(x \wedge H/L_+)$$

is compact in  $\text{SH}(H/H \cap N)$  for all  $L \leq H$ . So, since  $\text{ind} \cong \text{coind}$  preserves compacts it follows (using  $H/H \cap N \cong HN/N \hookrightarrow G/N$ ) that

$$\lambda^{N,G}(\text{coind}_H^G(x \wedge H/L_+))$$

is compact in  $\text{SH}(G/N)$  for all  $L \leq H$ . On the other hand, for any  $K \leq G$ , we have

$$\begin{aligned} \lambda^{N,G}(\text{ind}_H^G(x) \wedge G/K_+) &\cong \lambda^{N,G}(\text{ind}_H^G(x \wedge \text{res}_H^G(G/K_+))) \\ &\cong \bigoplus_{[g] \in H \backslash G/K} \lambda^{N,G}(\text{ind}_H^G(x \wedge (H/H^g \cap K)_+)) \end{aligned}$$

by the double-coset formula, which is compact. This proves  $\text{ind}_H^G(x) \in \mathcal{A}_{N,G}$ .  $\square$

6.5. *Remark.* Recall that a  $G$ -spectrum is said to be  $N$ -concentrated if it is local for the finite localization associated to the family  $\mathcal{F}[\not\supseteq N] := \{H \leq G \mid H \not\supseteq N\}$ . By definition, the geometric  $N$ -fixed point functor  $\tilde{\Phi}^N : \text{SH}(G) \rightarrow \text{SH}(G/N)$  is given by  $\tilde{\Phi}^N(x) := \lambda^N(\tilde{E}\mathcal{F}[\not\supseteq N] \wedge x)$ . Thus, if  $x \in \text{SH}(G)^c$  is  $N$ -concentrated then  $\lambda^N(x) \cong \tilde{\Phi}^N(x)$  is just the geometric  $N$ -fixed points of  $x$ , and is therefore compact in  $\text{SH}(G/N)$ . It follows that if  $x \in \text{SH}(G)^c$  is  $N$ -concentrated then  $x \in \mathcal{A}_{N,G}$ .

6.6. *Remark.* It will also be useful to recall (cf. [BS16, Sec. 2 (H)]) that restricting  $\lambda^N : \text{SH}(G) \rightarrow \text{SH}(G/N)$  to the subcategory of  $N$ -concentrated  $G$ -spectra produces an equivalence  $\lambda^N : \tilde{E}\mathcal{F}[\not\supseteq N] \wedge \text{SH}(G) \xrightarrow{\sim} \text{SH}(G/N)$ , with quasi-inverse given by  $x \mapsto \tilde{E}\mathcal{F}[\not\supseteq N] \wedge \text{infl}_{G/N}^G(x)$ .

6.7. **Lemma.** *Let  $K, N \trianglelefteq G$  be two normal subgroups of a finite group. Let  $x \in \mathcal{A}_{N,G}$  and assume that  $x$  is  $K$ -concentrated. Then  $\lambda^K(x) \in \mathcal{A}_{KN/K, G/K}$ .*

*Proof.* For any  $y \in \text{SH}(G/K)^c$ , we have

$$\begin{aligned} \lambda^{KN/K}(\lambda^K(x) \wedge y) &\cong \lambda^{KN/K} \lambda^K(x \wedge \text{infl}_{G/K}^G(y)) \\ &\cong \lambda^{KN}(x \wedge \text{infl}_{G/K}^G(y)) \end{aligned}$$

and  $x \wedge \text{infl}_{G/K}^G(y) \in \mathcal{A}_{N,G}$  and is also  $K$ -concentrated. So, it suffices to prove just that  $\lambda^{KN}(x)$  is compact in  $\text{SH}(G/KN)$ . Now compute in the opposite direction:  $\lambda^{KN}(x) \cong \lambda^{KN/N} \lambda^N(x)$ . Observe that if  $H \supseteq K$  then  $HN/N \supseteq KN/N$ . So, if  $H \leq G$  is such that  $HN/N \not\supseteq KN/N$  then  $H \not\supseteq K$  and hence  $G/H_+ \wedge x = 0$  since  $x$  is  $K$ -concentrated. In other words,  $E\mathcal{F}_+ \wedge x = 0$  where  $\mathcal{F} = \{H \leq G \mid HN/N \not\supseteq KN/N\}$ . But this family  $\mathcal{F}$  is the family “inflated” from the family of subgroups of  $G/N$  which do not contain  $KN/N$  (cf. [BS16, Thm 5.11] and [BS16, Cor. 4.5]). That is,  $\text{infl}_{G/N}^G(E\mathcal{F}[\not\supseteq KN/N]_+) \wedge x = 0$ , i.e.  $x \cong \text{infl}_{G/N}^G(\tilde{E}\mathcal{F}[\not\supseteq KN/N]) \wedge x$ . So,  $\lambda^N(x) \cong \lambda^N(\text{infl}_{G/N}^G(\tilde{E}\mathcal{F}[\not\supseteq KN/N]) \wedge x) \cong \tilde{E}\mathcal{F}[\not\supseteq KN/N] \wedge \lambda^N(x)$ . In other words, if  $x$  is  $K$ -concentrated then  $\lambda^N(x)$  is  $KN/N$ -concentrated. Hence  $\lambda^{KN}(x) \cong \lambda^{KN/N}(\lambda^N(x)) \cong \Phi^{KN/N}(\lambda^N(x))$  is compact *provided that*  $\lambda^N(x)$  is compact in  $\text{SH}(G/N)$ . But this is the case since  $x \in \mathcal{A}_{N,G}$  by assumption.  $\square$

**6.8. Proposition.** *Let  $G$  be a finite group and let  $N \trianglelefteq G$  be a normal subgroup. If  $N \subseteq \mathcal{O}^p(G)$  for some prime  $p$ , then  $\mathcal{P}(G, p, n) \in Z_{N, G}$  for all  $2 \leq n \leq \infty$ . If  $N \subseteq \mathcal{O}^p(G)$  for all primes  $p$ , then  $\mathcal{P}(G, 1) \in Z_{N, G}$ .*

*Proof.* Suppose  $N \subseteq \mathcal{O}^p(G)$  for some prime  $p$ . Consider the closed set  $\overline{\{\mathcal{P}(G, p, 2)\}} \subset \mathrm{Spc}(\mathrm{SH}(G)^c)$ . It is a Thomason closed set (cf. [BS16, Prop. 10.1]), and hence is the support of some  $X \in \mathrm{SH}(G)^c$ . We claim that  $\overline{G/H_+ \wedge X} = 0$  for all  $H \leq G$  such that  $H \not\supseteq N$ . Indeed, if  $\mathcal{P}(K, q, m) \in \mathrm{supp}(X) = \overline{\{\mathcal{P}(G, p, 2)\}}$  then it follows from [BS16, Cor. 6.4] and [BS16, Prop. 6.9] that  $q = p$  and that  $K$  is a  $p$ -subnormal subgroup of  $G$ . On the other hand, if  $\mathcal{P}(K, p, m) \in \mathrm{supp}(G/H_+)$  then  $K \leq_G H$  (by [BS16, Cor. 4.13]). Hence, if  $H \not\supseteq N$  then  $K \not\supseteq N$ , and so  $K \not\subseteq \mathcal{O}^p(G)$  by our hypothesis; that is,  $K$  is not a  $p$ -subnormal subgroup of  $G$  (by [BS16, Lem. 3.3]). We thus conclude that  $\mathrm{supp}(G/H_+) \cap \mathrm{supp}(X) = \emptyset$  (and hence  $G/H_+ \wedge X = 0$ ) if  $H \not\supseteq N$ . Hence  $E\mathcal{F}[\not\supseteq N]_+ \wedge X = 0$ . In other words,  $X$  is  $N$ -concentrated:  $X \cong X \wedge \tilde{E}\mathcal{F}[\not\supseteq N]$ . Hence  $X \in \mathcal{A}_{N, G}$  (cf. Rem. 6.5) so  $\mathcal{P}(G, p, 2) \in Z_{N, G}$ , which proves the first claim (cf. [BS16, Prop. 6.2]). Next, suppose  $N \subseteq \mathcal{O}^p(G)$  for all primes  $p$ . Then we can apply the same argument to the Thomason closed subset  $\overline{\{\mathcal{P}(G, 1)\}}$ . In this case, if a prime  $\mathcal{P}(K, q, m) \in \overline{\{\mathcal{P}(G, 1)\}}$  then  $K$  is a  $q$ -subnormal subgroup of  $G$ , and we use  $N \subseteq \mathcal{O}^q(G)$  to ensure that such a prime cannot be contained in  $\mathrm{supp}(G/H_+)$  when  $H \not\supseteq N$ . The argument is otherwise identical.  $\square$

**6.9. Proposition.** *Let  $G$  be a  $p$ -group and let  $1 \neq N \trianglelefteq G$  be a nontrivial normal subgroup. Then  $\mathcal{P}(G, p, n) \notin Z_{N, G}$  for all  $2 \leq n \leq \infty$ .*

*Proof.* For any  $2 \leq m < \infty$ , the subset  $\overline{\{\mathcal{C}_{p, m}\}} \subset \mathrm{Spc}(\mathrm{SH}^c)$  is a Thomason closed subset of the spectrum of the non-equivariant stable homotopy category (cf. [BS16, Cor. 10.5] or [Bal10a, Cor. 9.5(d)]). Hence, there exists  $X_{p, m} \in \mathrm{SH}^c$  such that  $\mathrm{supp}(X_{p, m}) = \overline{\{\mathcal{C}_{p, m}\}}$ . Then  $\mathrm{triv}_G(X_{p, m}) \in \mathrm{SH}(G)^c$  has support in  $\mathrm{Spc}(\mathrm{SH}(G)^c)$  equal to

$$\mathrm{supp}(\mathrm{triv}_G(X_{p, m})) = \{ \mathcal{P}(H, p, n) \mid n \geq m, p \text{ fixed}, H \leq G \}.$$

Suppose for a contradiction that  $\mathcal{P}(G, p, n) \in Z_{N, G}$  for some  $2 \leq n < \infty$ . Then since  $G$  is a  $p$ -group, every subgroup of  $G$  is a  $p$ -subnormal subgroup, and thus there is some  $m \geq n$  such that  $\mathrm{supp}(\mathrm{triv}_G(X_{p, m})) \subseteq Z_{N, G}$ . Indeed, we can take  $m = n + \log_p(|G|)$  for example (cf. [BS16, Cor. 8.3]). Hence,  $\mathrm{triv}_G(X_{p, m}) \in \mathcal{A}_{N, G}$ , so that  $\lambda^N(\mathrm{triv}_G(X_{p, m}))$  is a compact object of  $\mathrm{SH}(G/N)$ . We claim that this cannot be the case. Indeed, by the projection formula we have

$$\lambda^N(\mathrm{triv}_G(X_{p, m})) \cong \lambda^N(\mathrm{infl}_{G/N}^G(\mathrm{triv}_{G/N}(X_{p, m}))) \cong \lambda^N(\mathbb{1}) \otimes \mathrm{triv}_{G/N}(X_{p, m})$$

and the tom Dieck splitting theorem implies that  $\lambda^N(\mathbb{1})$  has

$$\Sigma_{G/N}^\infty E\mathcal{F}(N; G)/N_+$$

as a direct summand. Since restriction  $\mathrm{SH}(G/N) \rightarrow \mathrm{SH}$  preserves compact objects, we would then have that  $\Sigma^\infty BN_+ \wedge X_{p, m}$  is compact in  $\mathrm{SH}$ . But  $\mathcal{C}_{p, \infty} \in \mathrm{supp}(X_{p, m})$  means that  $X_{p, m} \notin \mathcal{C}_{p, \infty}$ ; i.e.  $H\mathbb{F}_{p*}(X_{p, m}) \neq 0$ . Since  $X_{p, m}$  is compact (hence dualizable) we similarly have non-vanishing of cohomology:  $H\mathbb{F}_p^*(X_{p, m}) \neq 0$ . Now  $H\mathbb{F}_p^*$  is a tensor-functor to the category of graded  $\mathbb{F}_p$ -modules. So if  $\Sigma^\infty BN_+ \wedge X_{p, m}$

were compact then

$$\begin{aligned} H\mathbb{F}_p^*(\Sigma^\infty BN_+ \wedge X_{p,m}) &\cong H\mathbb{F}_p^*(\Sigma^\infty BN_+) \otimes_{\mathbb{F}_p} H\mathbb{F}_p^*(X_{p,m}) \\ &\cong H^*(BN; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H\mathbb{F}_p^*(X_{p,m}) \end{aligned}$$

would be concentrated in finitely many degrees. Since  $H\mathbb{F}_p^*(X_{p,m}) \neq 0$ , this contradicts Proposition 3.1. This completes the proof that  $\mathcal{P}(G, p, n) \notin Z_{N,G}$  for all  $2 \leq n < \infty$ . Finally, let's prove that  $\mathcal{P}(G, p, \infty) \notin Z_{N,G}$ . By definition  $Z_{N,G}$  is a Thomason closed subset. Thus, by [BS16, Cor. 10.5] it is a union of irreducible closed sets  $\overline{\{\mathcal{P}(H, q, n)\}}$  for  $H \leq G$ ,  $q$  a prime, and  $1 \leq n < \infty$  finite. Thus,  $\mathcal{P}(G, p, \infty) \in Z_{N,G}$  necessarily implies that  $\mathcal{P}(G, p, n) \in Z_{N,G}$  for some finite  $n$ , which we have just shown is not possible.  $\square$

**6.10. Proposition.** *Let  $G$  be a finite group, and let  $N \trianglelefteq G$  be a normal subgroup. If  $N \not\subseteq O^p(G)$  then  $\mathcal{P}(G, p, n) \notin Z_{N,G}$  for all  $2 \leq n \leq \infty$ .*

*Proof.* For any fixed prime  $p$  and  $2 \leq m < \infty$ , consider the closed subset

$$Z_{p,m} := \{ \mathcal{P}(H, p, n) \mid n \geq m, p \text{ fixed}, H \leq G \} = \bigcup_{H \leq G} \overline{\{\mathcal{P}(H, p, m)\}}$$

of  $\text{Spc}(\text{SH}(G)^c)$  (cf. [BS16, Prop. 6.2]). By [BS16, Cor. 10.5], it is a Thomason closed subset. Hence, there exists  $X_{p,m} \in \text{SH}(G)^c$  such that  $\text{supp}(X_{p,m}) = Z_{p,m}$ . Now consider the family of subgroups  $\mathcal{F} := \mathcal{F}[\not\supseteq O^p(G)] = \{ H \leq G \mid H \not\supseteq O^p(G) \}$ . The set  $Z_{p,m}$  decomposes into two disjoint subsets:

$$\{ \mathcal{P}(H, p, n) \mid n \geq m, p \text{ fixed}, H \in \mathcal{F} \} \amalg \{ \mathcal{P}(H, p, n) \mid n \geq m, p \text{ fixed}, H \notin \mathcal{F} \}.$$

The first subset is closed; indeed, it is equal to

$$Z_{p,m} \cap \bigcup_{H \in \mathcal{F}} \text{supp}(G/H_+) = \bigcup_{H \in \mathcal{F}} \overline{\{\mathcal{P}(H, p, m)\}} =: Z_1.$$

On the other hand, the key reason for our choice of this particular family  $\mathcal{F}$  is that the right-hand subset is also closed. Indeed,

$$(6.11) \quad \{ \mathcal{P}(H, p, n) \mid n \geq m, p \text{ fixed}, H \notin \mathcal{F} \} = \bigcup_{H \supseteq O^p(G)} \overline{\{\mathcal{P}(H, p, m)\}} =: Z_2.$$

The inclusion  $\subseteq$  is evident. The point is that if  $\mathcal{P}(K, q, l) \in \overline{\{\mathcal{P}(H, p, m)\}}$ , then  $q = p$  and  $K$  is conjugate to a  $p$ -subnormal subgroup of  $H$ . But  $H \supseteq O^p(G)$  means that  $H$  is a  $p$ -subnormal subgroup of  $G$ . So  $K$  is conjugate to a  $p$ -subnormal subgroup of  $G$  which means that  $K^g \supseteq O^p(G)$  for some  $g \in G$  so that  $\mathcal{P}(K, q, l) = \mathcal{P}(K^g, p, l)$  is contained in the left-hand side of (6.11).

Thus,  $\text{supp}(X_{p,m}) = Z_1 \amalg Z_2$  is a decomposition into two disjoint closed sets. It follows from [Bal07, Thm. 2.11] that we have a decomposition  $X_{p,m} = x_1 \oplus x_2$  where  $\text{supp}(x_i) = Z_i$  for each  $i = 1, 2$ . In fact, one easily sees that  $x_1 \cong E\mathcal{F}_+ \wedge X_{p,m}$  and  $x_2 \cong \tilde{E}\mathcal{F} \wedge X_{p,m}$ . The key point is that these objects are compact in  $\text{SH}(G)$ .

Now, suppose for a contradiction that  $\mathcal{P}(G, p, n) \in Z_{N,G}$  for some  $2 \leq n < \infty$ . Then

$$\text{supp}(\tilde{E}\mathcal{F} \wedge X_{p,m}) \subset \overline{\{\mathcal{P}(G, p, n)\}} \subset Z_{N,G}$$

for  $m$  large enough, e. g., for  $m \geq n + \log_p(|G/O^p(G)|)$ . Then  $\tilde{E}\mathcal{F} \wedge X_{p,m} \in \mathcal{A}_{N,G}$  for  $m$  large enough. By Lemma 6.7,

$$\lambda^{O^p(G)}(\tilde{E}\mathcal{F} \wedge X_{p,m}) \cong \Phi^{O^p(G)}(\tilde{E}\mathcal{F} \wedge X_{p,m})$$

is contained in  $\mathcal{A}_{\mathrm{Op}(G)N/\mathrm{Op}(G),G/\mathrm{Op}(G)}$ . Moreover,

$$\tilde{E}\mathcal{F} \wedge X_{p,m} \cong \tilde{E}\mathcal{F} \wedge \mathrm{infl}_{G/\mathrm{Op}(G)}^G(\Phi^{\mathrm{Op}(G)}(\tilde{E}\mathcal{F} \wedge X_{p,m}))$$

by Remark 6.6. Thus  $\mathcal{P}(G, p, m) \in \mathrm{supp}(X_{p,m})$  implies that

$$\mathcal{P}(G, p, m) \in \mathrm{supp}(\mathrm{infl}_{G/\mathrm{Op}(G)}^G(\Phi^{\mathrm{Op}(G)}(\tilde{E}\mathcal{F} \wedge X_{p,m})))$$

and hence (by [BS16, Cor. 4.5]) that

$$\mathcal{P}(G/\mathrm{Op}(G), p, m) \in \mathrm{supp}(\Phi^{\mathrm{Op}(G)}(\tilde{E}\mathcal{F} \wedge X_{p,m})).$$

In summary, if  $\mathcal{P}(G, p, n) \in Z_{N,G}$  for some  $2 \leq n < \infty$  then  $\mathcal{P}(G/\mathrm{Op}(G), p, m) \in Z_{N\mathrm{Op}(G)/\mathrm{Op}(G),G/\mathrm{Op}(G)}$  for some  $m \geq n \geq 2$ . Our assumption that  $N \not\subseteq \mathrm{Op}(G)$  implies that  $N\mathrm{Op}(G)/\mathrm{Op}(G)$  is a nontrivial subgroup of the  $p$ -group  $G/\mathrm{Op}(G)$ , and hence we conclude by Proposition 6.9. Finally,  $\mathcal{P}_G(G, p, \infty) \in Z_{N,G}$  implies that  $\mathcal{P}_G(G, p, n) \in Z_{N,G}$  for some finite  $n$  (by the same argument as in the proof of Proposition 6.9) and so the full claim is proved.  $\square$

Finally, let us prove Theorem 6.2.

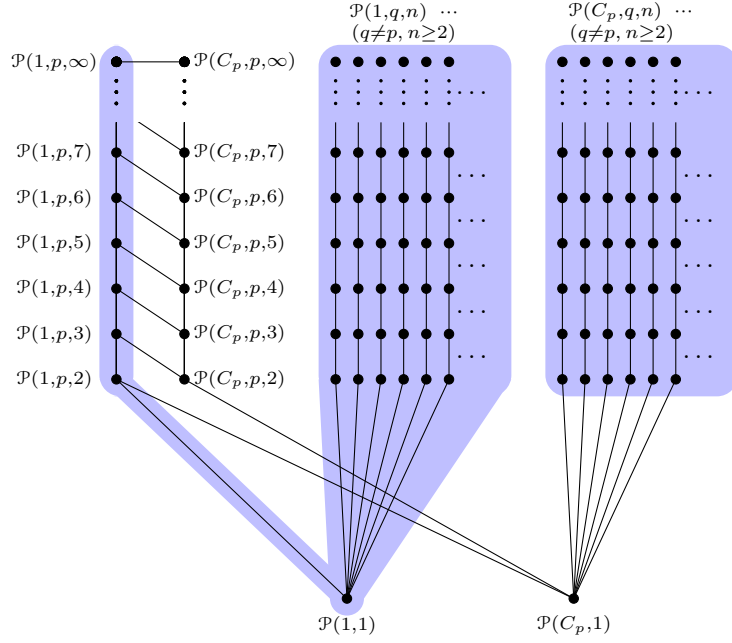
*Proof of Theorem 6.2.* By Proposition 6.4, it suffices to prove (a) in the case  $H = G$ , which is Proposition 6.8. On the other hand, Proposition 6.4 and Proposition 6.10 together prove (b) and the  $\Leftarrow$  direction of (c). Finally, the  $\Rightarrow$  direction of (c) follows from (a), since  $\mathcal{P}_G(H, 1) \in Z_{N,G}$  implies  $\mathcal{P}_G(H, p, 2) \in Z_{N,G}$  for all  $p$ .  $\square$

6.12. *Remark.* We proved in Proposition 3.2 that  $\mathrm{infl}_{G/N}^G$  does not satisfy GN-duality, except when  $N = 1$ . We can also deduce this (for  $G$  finite) from Theorem 6.2. Indeed, if  $\mathrm{infl}_{G/N}^G$  satisfied GN-duality then by the theorem,  $H \cap N \subset \mathrm{Op}^p(H)$  for all  $H \leq G$  which implies that  $N = 1$ . Indeed, if  $p \mid |N|$  then  $N$  would contain a subgroup  $H$  of order  $p$ , yielding the contradiction  $H = H \cap N \subset \mathrm{Op}^p(H) = 1$ .

6.13. *Example.* The compactness locus for  $G = N = C_p$ , the cyclic group of order  $p$ , is displayed in Figure 1 below. Note that the  $G$ -free spectra correspond to the irreducible component  $\{\mathcal{P}(1, 1)\} = \mathrm{supp}(G_+)$ .

6.14. *Example.* The compactness locus for  $G = D_{10}$ , the dihedral group of order 10, is displayed in Figure 2 (for  $N = G$ ) and in Figure 3 (for  $N = C_5$ ) on pages 30–31. (We have a complete understanding of the topology of the spectrum in this example since  $D_{10}$  is a group of square-free order; cf. [BS16, Thm 8.12].) The group  $D_{10}$  has a unique (normal) subgroup of order 5, and has five subgroups of order 2 (forming a single conjugacy class). Observe the different behavior at the primes 2 and 5 due to the fact that  $C_5$  is normal while the copies of  $C_2$  are not.

6.15. *Remark.* The Adams isomorphism is classically defined for  $N$ -free  $G$ -spectra (cf. Ex. 3.8). Note that in both cases of Ex. 6.14, even if we look  $p$ -locally, the compactness locus is larger than what is given by the  $N$ -free  $G$ -spectra. For  $N = G$ , the  $G$ -free spectra correspond to the irreducible closed subset  $\mathrm{supp}(G_+) = \overline{\{\mathcal{P}(1, 1)\}}$ , while for  $N = C_5$  the  $N$ -free  $G$ -spectra correspond to  $\mathrm{supp}(G_+) \cup \mathrm{supp}(G/C_{2+}) = \overline{\{\mathcal{P}(1, 1)\}} \cup \overline{\{\mathcal{P}(C_2, 1)\}}$ .

FIGURE 1. The compactness locus of  $\text{infl}_{G/N}^G$  for  $G = N = C_p$ .

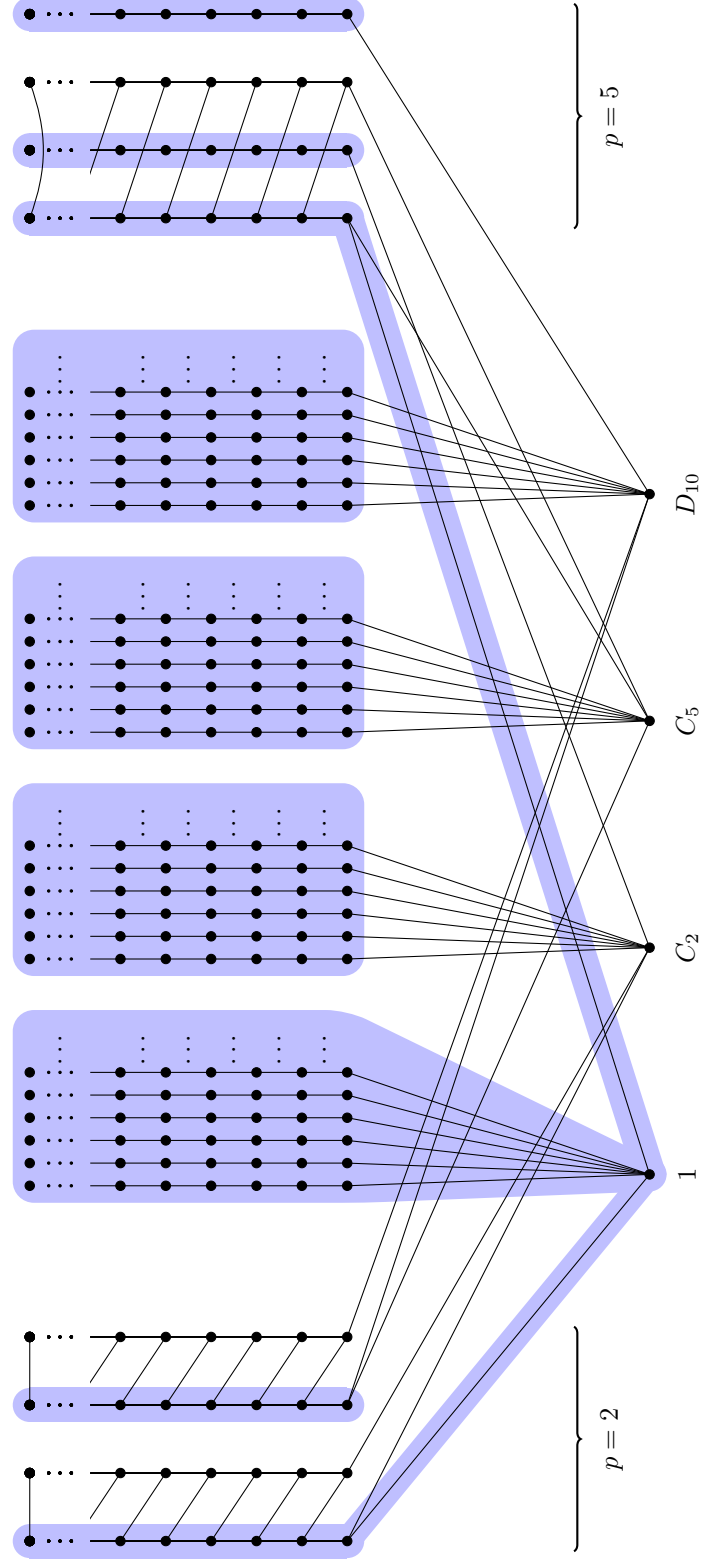
## 7. THE COMPACTNESS LOCUS IN ALGEBRAIC GEOMETRY

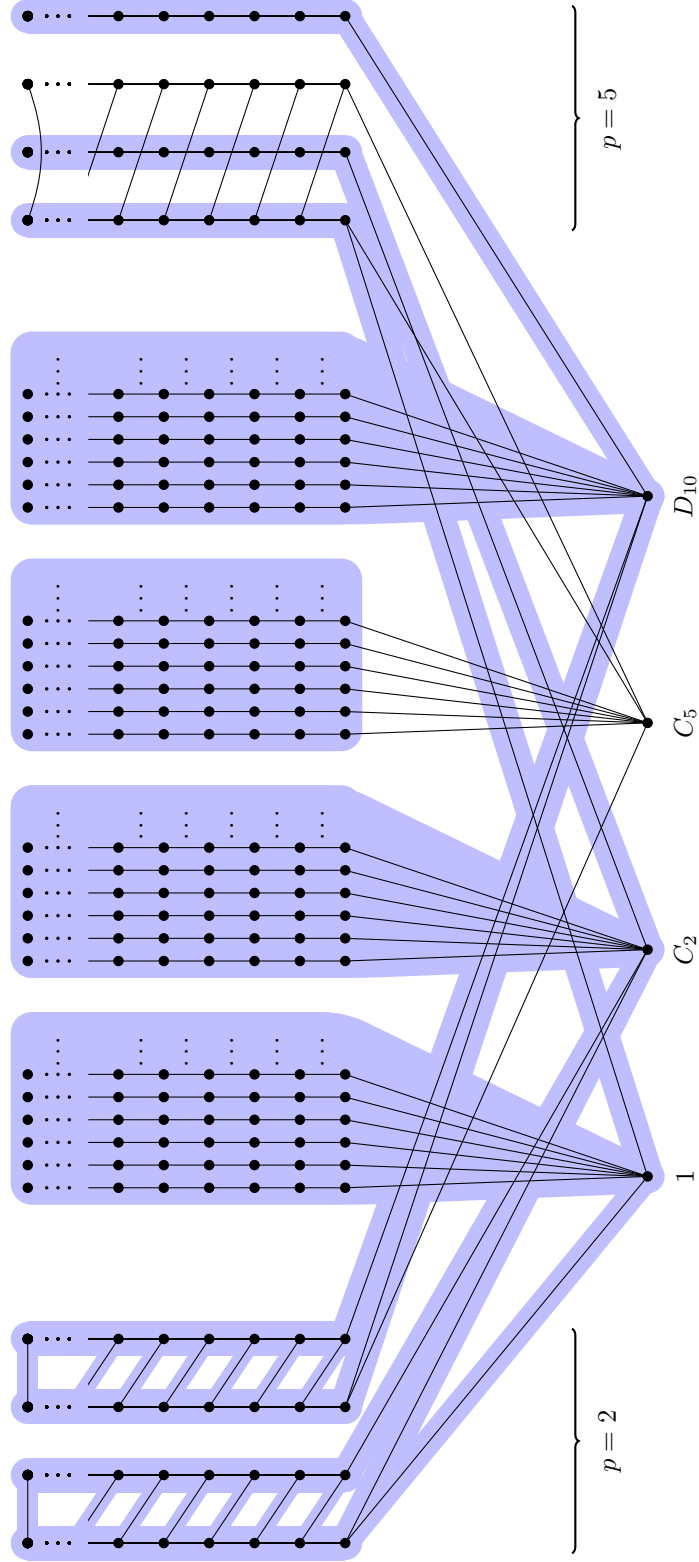
Our next goal is to provide a geometric description of the compactness locus for examples arising in algebraic geometry. Namely, if  $f : X \rightarrow Y$  is a morphism of quasi-compact and quasi-separated schemes, then the compactness locus of the derived pull-back  $f^* : D_{\text{Qcoh}}(Y) \rightarrow D_{\text{Qcoh}}(X)$  is a categorically defined subset of the domain scheme:

$$Z_f \subset X \cong \text{Spc}(D_{\text{Qcoh}}(X)^c).$$

We would like to obtain a scheme-theoretic description of  $Z_f \subset X$  in terms of the morphism  $f$ .

**7.1. Remark.** A natural first guess is that  $Z_f$  might describe something like the “locus of points where Grothendieck-Neeman duality is satisfied.” However, there are two subtleties which indicate that one should be careful. Firstly, although satisfying GN-duality is local in the target, it is not local in the source. Indeed, Proposition 5.3 provides a description of the compactness locus for an open immersion (Example 5.6); in particular, an open immersion only satisfies GN-duality if it is the inclusion of a set which is both open and Thomason closed (i.e. a union of connected components in the noetherian case). In fact, Lipman and Neeman [LN07] have made a detailed study of those morphisms which satisfy GN-duality. They show, for example, that a separated morphism of finite type  $f : X \rightarrow Y$  between noetherian schemes satisfies GN-duality if and only if  $f$  is proper and perfect. Although perfection is a local notion, properness is not. Nevertheless, one might then guess that the compactness locus coincides with the perfect locus when the morphism  $f$  is proper. However, again one must be careful. The perfect locus of  $f$  is closed under generalization, but by definition the compactness locus of  $f^*$  is

FIGURE 2. The compactness locus of  $\text{infl}_{G/N}^G$  for  $G = N = D_{10}$ .

FIGURE 3. The compactness locus of  $\text{infl}_{G/N}^G$  for  $G = D_{10}$ ,  $N = C_5$ .



closed under specialization. Thus, at first glance, the relationship (if any) between the categorically defined compactness locus of  $f^*$  and the scheme-theoretic perfect locus of  $f$  is somewhat mysterious. In Theorem 7.18 below, we'll prove that in fact (for  $f$  proper), the compactness locus of  $f^*$  is the largest specialization closed subset of  $X$  contained in the perfect locus of  $f$ . In other words, a point  $x \in X$  is contained in the compactness locus if and only if its closure  $\overline{\{x\}}$  is contained in the perfect locus.

In any case, let us begin our study in earnest. In order to avoid additional technicalities, we will mostly restrict our study to finite type morphisms between noetherian schemes, although some results can be generalized. We begin by recalling some terminology.

**7.2. Terminology.** Let  $X$  be a quasi-compact and quasi-separated scheme. Recall that the compact objects of  $D_{\text{Qcoh}}(X)$  are the perfect complexes, i.e. those complexes which are locally quasi-isomorphic to bounded complexes of finitely generated free modules. This is equivalent to being pseudo-coherent and of finite tor-dimension (cf. [TT90, Sec. 2]). For a morphism  $f : X \rightarrow Y$ , we will be interested in the relationship between  $\mathcal{E} \in D_{\text{Qcoh}}(X)$  being perfect,  $\mathbb{R}f_*(\mathcal{E}) \in D_{\text{Qcoh}}(Y)$  being perfect, and  $\mathcal{E}$  being perfect relative to  $f$ , i.e. regarded as a complex of  $f^{-1}\mathcal{O}_Y$ -modules (see [BGI71, Exp. III, Sec. 4] for the precise definition). A morphism of finite type  $f : X \rightarrow Y$  is said to be perfect if  $\mathcal{O}_X$  is perfect relative to  $f$ .

**7.3. Lemma.** *Let  $f : X \rightarrow Y$  be a morphism of finite type between noetherian schemes. For a point  $x \in X$ , the following two conditions are equivalent:*

- (a) *There exists an open neighbourhood  $U$  of  $x$  such that  $U \hookrightarrow X \rightarrow Y$  is a perfect morphism.*
- (b) *The local ring  $\mathcal{O}_{X,x}$  has finite tor-dimension when regarded as an  $\mathcal{O}_{Y,f(x)}$ -module.*

*Proof.* First we prove (a) $\Rightarrow$ (b). Let  $j : U \hookrightarrow X$  be an open neighbourhood of  $x$  such that  $f \circ j : U \rightarrow Y$  is perfect. By definition this means that  $\mathcal{O}_U$  has finite tor-dimension as a complex of  $(f \circ j)^{-1}\mathcal{O}_Y$ -modules. Taking the stalk at  $x$  it follows that  $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x}$  has finite tor-dimension as a complex of  $((f \circ j)^{-1}\mathcal{O}_Y)_x \cong \mathcal{O}_{Y,f(x)}$ -modules (using e.g. the characterization of tor-dimension in terms of the existence of bounded resolutions by flat modules). Conversely, let's prove (b) $\Rightarrow$ (a). Suppose  $\mathcal{O}_{X,x}$  has finite tor-dimension as an  $\mathcal{O}_{Y,f(x)}$ -module, say  $h := \text{tor-dim}_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ . Since  $f$  is of finite type, there exists an open neighbourhood  $j : W \hookrightarrow X$  of  $x$  such that  $f \circ j : W \rightarrow Y$  factors as a closed immersion  $i : W \hookrightarrow X'$  followed by a smooth morphism  $g : X' \rightarrow Y$ . (Moreover, we may assume without loss of generality that  $i : W \hookrightarrow X'$  is a closed immersion of affine schemes.) If  $g$  is smooth of relative dimension  $d$ , the argument in the proof of [BGI71, Prop. III.3.6(ii)] implies that  $\text{tor-dim}_{\mathcal{O}_{X',x}}(i_*\mathcal{O}_W)_x \leq h + d$ . Now the set of points  $x' \in X'$  such that  $\text{tor-dim}_{\mathcal{O}_{X',x'}}(i_*\mathcal{O}_W)_{x'} \leq h + d$  is an open subset of  $X'$  (cf. [BS13, Cor. 9.4.7]). So

$$U := \{ w \in W \mid \text{tor-dim}_{\mathcal{O}_{X',w}}(i_*\mathcal{O}_W)_w \leq h + d \}$$

is an open neighbourhood of  $x$  in  $W$  (and hence in  $X$ ). Moreover, for  $w \in U$  we have

$$\begin{aligned} \operatorname{tor-dim}_{\mathcal{O}_{Y,f(w)}} \mathcal{O}_{U,w} &= \operatorname{tor-dim}_{\mathcal{O}_{Y,f(w)}} \mathcal{O}_{W,w} \\ &= \operatorname{tor-dim}_{\mathcal{O}_{Y,f(w)}} (i_* \mathcal{O}_W)_w \\ &\leq \operatorname{tor-dim}_{\mathcal{O}_{X',w}} (i_* \mathcal{O}_W)_w \\ &\leq h + d \end{aligned}$$

where the first inequality comes from the fact that smooth morphisms are perfect. It follows (cf. [BGI71, Prop. III.3.3]) that  $U \hookrightarrow X \rightarrow Y$  has finite tor-dimension  $\leq h + d$ . Since it is automatically pseudo-coherent, we conclude that it is a perfect morphism.  $\square$

**7.4. Definition.** Let  $f : X \rightarrow Y$  be a morphism of finite type between noetherian schemes. The *perfect locus* of  $f$  is the set

$$\mathcal{P}_f := \{x \in X \mid \mathcal{O}_{X,x} \text{ has finite tor-dimension as an } \mathcal{O}_{Y,f(x)}\text{-module}\}.$$

It is an open subset of  $X$  and the morphism  $f$  is perfect precisely when  $\mathcal{P}_f = X$  (cf. Lemma 7.3).

**7.5. Terminology.** A functor  $F : D(\mathcal{A}) \rightarrow D(\mathcal{B})$  between two derived categories (or two subcategories thereof) is said to be *bounded above* if there exists an integer  $d$  such that for any object  $\mathcal{E}$  and integer  $m$ ,  $H^i(\mathcal{E}) = 0$  for  $i > m$  implies  $H^i(F(\mathcal{E})) = 0$  for  $i > m + d$ . Similarly, the functor is *bounded below* if there exists an integer  $d$  such that  $H^i(\mathcal{E}) = 0$  for  $i < m$  implies that  $H^i(F(\mathcal{E})) = 0$  for  $i < m - d$ . We say the functor is *bounded* if it is both bounded above and bounded below.

**7.6. Proposition.** Let  $f : X \rightarrow Y$  be a morphism of quasi-compact, quasi-separated schemes. If  $\mathcal{E} \in \mathcal{A}_f$  then  $\operatorname{hom}(\mathcal{E}, f^!(-)) : D_{\operatorname{Qcoh}}(Y) \rightarrow D_{\operatorname{Qcoh}}(X)$  is bounded.

*Proof.* The functor  $f^!$  is bounded below (cf. [Lip09, Thm 4.1]) and it is straightforward to see that  $\operatorname{hom}(\mathcal{E}, -) \cong \Delta \mathcal{E} \otimes -$  is bounded for any perfect  $\mathcal{E}$  (since the perfect complex  $\Delta \mathcal{E}$  is globally quasi-isomorphic to a bounded complex of flat modules). Thus, the functor  $\operatorname{hom}(\mathcal{E}, f^!(-))$  is bounded below for any perfect  $\mathcal{E}$ ; the key is to prove that it is bounded above when  $\mathcal{E} \in \mathcal{A}_f$ . To prove this, we will adapt the methods of [LN07, Sect. 3–4]. By [BvdB03, Thm. 3.1.1],  $D_{\operatorname{Qcoh}}(X)$  is generated by a single perfect complex  $\mathcal{S} \in D_{\operatorname{Qcoh}}(X)$ ; that is, there exists a perfect complex  $\mathcal{S}$  such that if  $\mathcal{G} \in D_{\operatorname{Qcoh}}(X)$  then

$$\mathcal{G} \neq 0 \text{ in } D_{\operatorname{Qcoh}}(X) \implies \operatorname{Hom}(\mathcal{S}[n], \mathcal{G}) \neq 0 \text{ for some } n \in \mathbb{Z}.$$

More precisely, by [LN07, Thm. 4.2], if  $\mathcal{S}$  is such a perfect generator, there exists an integer  $A = A(\mathcal{S})$  such that for any  $\mathcal{G} \in D_{\operatorname{Qcoh}}(X)$  and  $j \in \mathbb{Z}$ ,

$$(7.7) \quad H^j(\mathcal{G}) \neq 0 \implies \operatorname{Hom}(\mathcal{S}[n], \mathcal{G}) \neq 0 \text{ for some } n \leq A - j.$$

On the other hand,  $\mathcal{E} \in \mathcal{A}_f$  implies that  $f_*(\mathcal{E} \otimes \mathcal{S})$  is perfect, and hence is  $a$ -locally projective for some  $a \in \mathbb{Z}$  (cf. [LN07, p. 218]). Hence, by [LN07, Lem. 3.2], there exists  $s = s(Y) > 0$  such that

$$\operatorname{Hom}(f_*(\mathcal{E} \otimes \mathcal{S}), \mathcal{F}[-n]) = 0$$

for any  $\mathcal{F} \in D_{\operatorname{Qcoh}}(Y)$  for which  $H^j(\mathcal{F}) = 0$  for  $j > a - s - n$ . Then let's prove that  $\operatorname{hom}(\mathcal{E}, f^!(-))$  is bounded above. To this end, consider  $\mathcal{F} \in D_{\operatorname{Qcoh}}(Y)$  and suppose

that  $H^j(\mathcal{F}) = 0$  for  $j > m$ . If  $j \geq m + A + s - a$  then for any  $n \leq A - j$  we have that  $a - s - n \geq a - s - A + j \geq m$  and hence

$$\mathrm{Hom}(\mathcal{S}[n], \mathrm{hom}(\mathcal{E}, f^! \mathcal{F})) \cong \mathrm{Hom}(f_*(\mathcal{E} \otimes \mathcal{S}), \mathcal{F}[-n]) = 0.$$

Hence by (7.7), we conclude that  $H^j(\mathrm{hom}(\mathcal{E}, f^! \mathcal{F})) = 0$ . This establishes that  $\mathrm{hom}(\mathcal{E}, f^!(-))$  is bounded above.  $\square$

**7.8. Lemma.** *Let  $f : X \rightarrow Y$  be a morphism of quasi-compact, quasi-separated schemes. Let  $V \subset Y$  be a quasi-compact open subset, and set  $f_V := f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ . If  $\mathcal{E} \in \mathcal{A}_f$  then  $\mathcal{E}|_{f^{-1}(V)} \in \mathcal{A}_{f_V}$ .*

*Proof.* Let  $\mathcal{E} \in \mathcal{A}_f$  and consider  $\mathcal{E}|_{f^{-1}(V)} \in \mathrm{D}_{\mathrm{Qcoh}}(f^{-1}(V))^c$ . We claim that

$$\mathbb{R}(f_V)_*(\mathcal{E}|_{f^{-1}(V)} \otimes \mathcal{F})$$

is compact in  $\mathrm{D}_{\mathrm{Qcoh}}(V)$  for any  $\mathcal{F} \in \mathrm{D}_{\mathrm{Qcoh}}(f^{-1}(V))^c$ . By the Thomason-Neeman localization theorem (cf. [Nee01, Cor. 4.5.14, Rem. 4.5.15]),  $\mathcal{F} \oplus \Sigma \mathcal{F} \simeq \mathcal{G}|_{f^{-1}(V)}$  for some  $\mathcal{G} \in \mathrm{D}_{\mathrm{Qcoh}}(X)^c$ , and it suffices to prove that

$$\mathbb{R}(f_V)_*(\mathcal{E}|_{f^{-1}(V)} \otimes \mathcal{G}|_{f^{-1}(V)}) \simeq \mathbb{R}(f_V)_*((\mathcal{E} \otimes \mathcal{F})_{f^{-1}(V)})$$

is compact. By flat base change we have

$$\mathbb{R}(f_V)_*((\mathcal{E} \otimes \mathcal{G})|_{f^{-1}(V)}) \simeq (\mathbb{R}f_*(\mathcal{E} \otimes \mathcal{G}))|_V$$

and the latter is compact since restriction to an open preserves compacts and  $\mathcal{E} \in \mathcal{A}_f$  by hypothesis.  $\square$

**7.9. Proposition.** *Let  $f : X \rightarrow Y$  be a morphism of quasi-compact, quasi-separated schemes. If  $\mathcal{E} \in \mathcal{A}_f$  then  $\mathcal{E}$  has finite tor-dimension as a complex of  $f^{-1}\mathcal{O}_Y$ -modules.*

*Proof.* Let  $Y = \bigcup V_i$  be an open affine cover and set  $f_i : f^{-1}(V_i) \rightarrow V_i$ . By Lemma 7.8, if  $\mathcal{E} \in \mathcal{A}_f$  then  $\mathcal{E}|_{f^{-1}(V_i)} \in \mathcal{A}_{f_i}$ . On the other hand, if  $\mathcal{E}|_{f^{-1}(V_i)}$  has finite tor-dimension as a complex of  $f_i^{-1}\mathcal{O}_{V_i} \simeq f^{-1}\mathcal{O}_Y|_{f^{-1}(V_i)}$ -modules for all  $i$ , then  $\mathcal{E}$  has finite tor-dimension as a complex of  $f^{-1}\mathcal{O}_Y$ -modules. Thus, the problem is local in the base, and we can assume without loss of generality that  $Y = \mathrm{Spec}(A)$  is affine.

Now to show that  $\mathcal{E}$  has finite tor-dimension as complex of  $f^{-1}(\mathcal{O}_Y)$ -modules, it suffices to check that  $\mathcal{E}|_U$  has finite tor-dimension as a complex of  $(f^{-1}\mathcal{O}_Y)|_U \simeq (f \circ j)^{-1}(\mathcal{O}_Y)$ -modules for any open affine  $j : U \hookrightarrow X$ , say  $U = \mathrm{Spec}(B)$ . Moreover, since  $f \circ j$  is a morphism of affine schemes, we just need to prove that  $(f \circ j)_*(\mathcal{E}|_U)$  has finite tor-dimension as a complex of  $\mathcal{O}_Y$ -modules (i.e. the complex of  $B$ -modules  $\mathcal{E}|_U$  has finite tor-dimension when regarded as a complex of  $A$ -modules).

Then observe that for any  $\mathcal{G} \in \mathrm{Qcoh}(Y)$  and  $j \in \mathbb{Z}$  we have

$$\begin{aligned} \mathrm{Ext}_{\mathcal{O}_Y}^j((f \circ j)_*(\mathcal{E}|_U), \mathcal{G}) &= \mathrm{Hom}_{\mathrm{D}(Y)}((f \circ j)_*(\mathcal{E}|_U), \mathcal{G}[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}(Y)}(\mathbb{R}(f \circ j)_*(\mathcal{E}|_U), \mathcal{G}[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}(Y)}(\mathbb{R}f_*\mathbb{R}j_*(\mathcal{E}|_U), \mathcal{G}[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{R}j_*(\mathcal{E}|_U), f^!\mathcal{G}[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{E} \otimes \mathbb{R}j_*\mathcal{O}_U, f^!\mathcal{G}[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{R}j_*\mathcal{O}_U, \mathrm{hom}(\mathcal{E}, f^!\mathcal{G})[j]). \end{aligned}$$

Applying [LN07, Lem. 3.3] to  $j : U \hookrightarrow X$ , there exists  $t = t_U > 0$  such that for any  $a$ -locally projective  $\mathcal{F} \in \mathrm{D}_{\mathrm{Qcoh}}(U)$ ,  $\mathbb{R}j_*(\mathcal{F}) \in \mathrm{D}_{\mathrm{Qcoh}}(X)$  is  $(a - t)$ -locally projective.

Now,  $\mathcal{O}_U$  is 0-locally projective, so then  $\mathbb{R}j_*(\mathcal{O}_U)$  is  $-t$ -locally projective. Then by [LN07, Lem. 3.2], there exists  $s = s(X) > 0$  such that

$$\mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{R}j_*\mathcal{O}_U, \mathrm{hom}(\mathcal{E}, f^!(\mathcal{G}))[j]) = 0$$

for any  $\mathcal{G} \in \mathrm{D}_{\mathrm{Qcoh}}(Y)$  with the property that  $H^i(\mathrm{hom}(\mathcal{E}, f^!\mathcal{G}))[j] = 0$  for all  $i > -t - s$ . On the other hand, by Proposition 7.6, we know that  $\mathrm{hom}(\mathcal{E}, f^!(-))$  is bounded since  $\mathcal{E} \in \mathcal{A}_f$  by assumption. So there exists  $m = m_{\mathcal{E}} \in \mathbb{Z}$  such that if  $\mathcal{G} \in \mathrm{D}_{\mathrm{Qcoh}}(Y)$  has the property that  $H^i(\mathcal{G}) = 0$  for  $i > n$ , then  $H^i(\mathrm{hom}(\mathcal{E}, f^!\mathcal{G})) = 0$  for all  $i > n + m$ . Thus, setting  $j_0 := m + t + s$  we conclude that

$$\mathrm{Ext}_{\mathcal{O}_Y}^j((f \circ j)_*(\mathcal{E}|_U), \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{R}i_*\mathcal{O}_U, \mathrm{hom}(\mathcal{E}, f^!\mathcal{G}))[j] = 0$$

for all  $j > j_0$  and  $\mathcal{G} \in \mathrm{Qcoh}(Y)$ . It follows that the bounded complex  $(f \circ j)_*(\mathcal{E}|_U)$  has finite projective dimension, hence has finite flat dimension (i.e. is isomorphic to a bounded complex of flat modules) which completes the proof.  $\square$

**7.10. Lemma.** *Let  $R$  be a commutative noetherian local ring and let  $M$  and  $N$  be complexes of  $R$ -modules. Suppose that  $M$  is bounded below and homologically finite, and that  $N$  is perfect. If  $N \neq 0$  in  $\mathrm{D}(R)$  and  $M \otimes_R^{\mathbb{L}} N$  is perfect then  $M$  is perfect.*

*Proof.* Over a noetherian ring, a bounded below, homologically finite complex is perfect if and only if it has finite projective dimension if and only if it has finite flat dimension, in which case the projective and flat dimensions coincide (cf. [AF91, Cor. 2.10.F]). Moreover, when  $R$  is local, its flat dimension can be computed as

$$(7.11) \quad \mathrm{fd}_R(M) = \sup \{ i \in \mathbb{Z} \mid H_i(k \otimes_R^{\mathbb{L}} M) \neq 0 \}.$$

This is mentioned, e.g., in [DGI06], and can be obtained by computing the derived tensor-product using a minimal free resolution of  $M$ . Now, if  $M \otimes_R^{\mathbb{L}} N$  is perfect then evidently  $\mathrm{fd}_R(M \otimes_R^{\mathbb{L}} N) < \infty$ , which implies that

$$(7.12) \quad H_i(k \otimes_R^{\mathbb{L}} (M \otimes_R^{\mathbb{L}} N)) = 0 \quad \text{for } i \gg 0.$$

Next note that since  $R$  is local, a perfect complex  $N$  is zero in  $\mathrm{D}(R)$  if and only if its homological support  $\mathrm{supp}_R(N) \neq \emptyset$  if and only if  $\mathrm{supp}_R(N)$  contains the closed point if and only if  $\mathrm{supp}_k(k \otimes_R^{\mathbb{L}} N) \neq \emptyset$  if and only if  $k \otimes_R^{\mathbb{L}} N \neq 0$  in  $\mathrm{D}(k)$ . So  $H_*(k \otimes_R^{\mathbb{L}} N) \neq 0$ . Hence, from

$$\begin{aligned} H_*(k \otimes_R^{\mathbb{L}} (M \otimes_R^{\mathbb{L}} N)) &\cong H_*((k \otimes_R^{\mathbb{L}} M) \otimes_k (k \otimes_R^{\mathbb{L}} N)) \\ &\cong H_*(k \otimes_R^{\mathbb{L}} M) \otimes_k H_*(k \otimes_R^{\mathbb{L}} N) \end{aligned}$$

we see that (7.12) implies  $H_*(k \otimes_R^{\mathbb{L}} M)$  must also be bounded above and thus  $M$  has finite projective dimension by (7.11).  $\square$

**7.13. Corollary.** *Let  $R$  be a commutative noetherian local ring and  $I \subset R$  an ideal. Let  $K(J)$  denote the Koszul complex associated to a nonzero ideal  $J \subset R$ . If  $R/I \otimes_R^{\mathbb{L}} K(J)$  is compact in  $\mathrm{D}(R)$  then  $R/I$  is compact in  $\mathrm{D}(R)$ .*

**7.14. Proposition.** *Let  $i : Z \hookrightarrow X$  be a closed immersion of a noetherian scheme  $X$ . Then the compactness locus  $Z_i$  is contained in the perfect locus  $\mathcal{P}_i$ .*

*Proof.* It follows from Lemma 7.8 that the problem is local in the base, so it suffices to prove the claim for a closed immersion of affine schemes. So let  $R$  be a commutative noetherian ring,  $I \subset R$  an ideal and set  $f := \mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}(R)$ . Let  $J$

be any ideal of  $R$  containing  $I$  and let  $\bar{J}$  denote the corresponding ideal of  $R/I$ . Now  $f^*(K_R(J)) \cong K_{R/I}(\bar{J})$ , and hence

$$f_*(K_{R/I}(\bar{J})) \cong R/I \otimes_R^{\mathbb{L}} K_R(J)$$

by the projection formula. Since  $\text{supp}(K_{R/J}(\bar{J})) = V(\bar{J})$ , we have that  $V(\bar{J}) \subset Z_f$  if and only if  $f_*(K_{R/I}(\bar{J}))$  is compact in  $D(R)$  if and only if  $R/I \otimes_R^{\mathbb{L}} K_R(J)$  is compact in  $D(R)$ . This is the case if and only if

$$((R/I) \otimes_R^{\mathbb{L}} K_R(J))_{\mathfrak{q}} \simeq R_{\mathfrak{q}}/I_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}}^{\mathbb{L}} K_{R_{\mathfrak{q}}}(J_{\mathfrak{q}})$$

is compact in  $D(R_{\mathfrak{q}})$  for all  $\mathfrak{q} \in V(J)$ . Invoking Corollary 7.13, we conclude that if  $V(J) \subset Z_f \subset V(I)$  then  $R_{\mathfrak{q}}/I_{\mathfrak{q}}$  is compact in  $D(R_{\mathfrak{q}})$  for all  $\mathfrak{q} \in V(J)$ , which means by definition that  $V(J) \subset \mathcal{P}_f$ . Since  $Z_f$  is closed under specialization, it follows that  $Z_f \subset \mathcal{P}_f$ .  $\square$

**7.15. Theorem.** *Let  $f : X \rightarrow Y$  be a morphism of finite type between noetherian schemes. Then the compactness locus of  $f^* : D_{\text{Qcoh}}(Y) \rightarrow D_{\text{Qcoh}}(X)$  is contained in the perfect locus of  $f$ .*

*Proof.* If  $x \in Z_f$  is a point in the compactness locus, then there exists a complex  $\mathcal{E} \in \mathcal{A}_f$  with  $\text{supp}(\mathcal{E}) = \overline{\{x\}}$ . By Proposition 7.9,  $\mathcal{E}$  has finite tor-dimension as a complex of  $f^{-1}\mathcal{O}_Y$ -modules. Since  $f$  is of finite type, there is an open affine neighbourhood  $j : U \hookrightarrow X$  of  $x$  such that  $f \circ j$  factors as a closed immersion  $i : U \hookrightarrow X'$  followed by a smooth morphism  $g : X' \rightarrow Y$ . Now,  $j^*(\mathcal{E}) \in D(U)^c$  has finite tor-dimension relative to  $f \circ j$ , hence [BGI71, p. 246, Prop. 3.6] implies that  $\mathbb{R}i_*(j^*\mathcal{E})$  is perfect. Moreover, since  $D_{\text{Qcoh}}(U)$  is generated by  $\mathbb{1}$  (since  $U$  is affine),  $\mathbb{R}i_*(j^*\mathcal{E})$  perfect implies  $j^*\mathcal{E} \in \mathcal{A}_i$ . Furthermore, since  $\text{supp}_U(j^*\mathcal{E}) = U \cap \text{supp}(\mathcal{E}) \ni x$ , we conclude that  $x \in Z_i$ . Hence by Proposition 7.14,  $x$  is contained in the perfect locus  $\mathcal{P}_i$  of the closed immersion  $i$ . Since  $g$  is smooth (hence perfect), it follows that  $x \in \mathcal{P}_{g \circ i}$ . Thus  $x \in \mathcal{P}_{f \circ j} = \mathcal{P}_f \cap \mathcal{P}_j = U \cap \mathcal{P}_f$ .  $\square$

**7.16. Remark.** Since the compactness locus is closed under specialization, Theorem 7.15 implies that it is contained in the largest specialization closed subset of  $X$  which is contained in the perfect locus, namely the union of all closed subsets of  $X$  which are contained in the perfect locus. In Theorem 7.18, we will show that this inclusion is in fact an equality for proper morphisms. However, it is not an equality in general. For example, an open immersion is perfect, hence the largest specialization closed subset contained in the perfect locus is just  $X$  itself, but we saw in Example 5.6 that the compactness locus is usually much smaller.

**7.17. Lemma.** *Let  $f : X \rightarrow Y$  be a proper morphism of noetherian schemes. Let  $\mathcal{E} \in D_{\text{Qcoh}}(X)^c$  be a compact object and assume that  $\mathcal{E}$  has finite tor-dimension as a complex of  $f^{-1}\mathcal{O}_Y$ -modules. Then  $\mathbb{R}f_*(\mathcal{E})$  is a compact object in  $D_{\text{Qcoh}}(Y)$ .*

*Proof.* The argument is essentially contained in [Sta16, Tag 08EV]. We sketch the proof for convenience. Since proper morphisms of noetherian schemes preserve pseudo-coherent complexes (see e.g. [BGI71, p. 236, Thm. 2.2] or [Lip09, Sec. 4.3]), it suffices to show that  $\mathbb{R}f_*\mathcal{E}$  has finite tor-dimension as a complex of  $\mathcal{O}_Y$ -modules. Since  $Y$  is quasi-separated, an object  $\mathcal{F} \in D_{\text{Qcoh}}(Y)$  has tor-amplitude in  $[a, b]$  if and only if  $H^i(\mathcal{F} \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \mathcal{G}) = 0$  for all  $i \notin [a, b]$  and  $\mathcal{G} \in \text{Qcoh}(Y)$ . (The point being that we need only check this condition for quasi-coherent sheaves  $\mathcal{G}$  rather

than for all  $\mathcal{O}_Y$ -modules; see e.g. [Sta16, Tag 08EA].) Then observe that for any  $\mathcal{G} \in \mathrm{Qcoh}(Y)$ , we have

$$\mathbb{R}f_*\mathcal{E} \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \mathcal{G} \simeq \mathbb{R}f_*(\mathcal{E} \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathbb{L}f^*\mathcal{G}) \simeq \mathbb{R}f_*(\mathcal{E} \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbb{L}} f^{-1}\mathcal{G}).$$

By assumption,  $\mathcal{E}$  has finite tor-dimension as an object of  $D(f^{-1}\mathcal{O}_Y)$ . Hence the complex  $\mathcal{E} \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbb{L}} f^{-1}\mathcal{G}$  has cohomology sheaves in a given finite range, say  $[a, b]$ . Then its image under  $\mathbb{R}f_*$  has cohomology in the range  $[a, b + d]$  for some integer  $d$  depending on  $Y$  but not on  $\mathcal{G}$  (cf. [Lip09, Prop. 3.9.2]). So there is a universal bound for the cohomology

$$H^i(\mathbb{R}f_*\mathcal{E} \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \mathcal{G})$$

for all quasi-coherent sheaves  $\mathcal{G} \in \mathrm{Qcoh}(Y)$ , independent of  $\mathcal{G}$ , and we conclude that  $\mathbb{R}f_*(\mathcal{E})$  has finite tor-dimension.  $\square$

**7.18. Theorem.** *Let  $f : X \rightarrow Y$  be a proper morphism of noetherian schemes. Then the compactness locus of  $f^* : D_{\mathrm{Qcoh}}(Y) \rightarrow D_{\mathrm{Qcoh}}(X)$  coincides with the largest specialization closed subset of  $X$  contained in the perfect locus of  $f$ . In other words, a point  $x \in X$  is contained in the compactness locus of  $f^*$  if and only if  $\overline{\{x\}}$  is contained in the perfect locus of  $f$ .*

*Proof.* We will first show that if a closed subset  $V \subset X$  is contained in the perfect locus  $\mathcal{P}_f$  then it is contained in the compactness locus  $Z_f$ . Indeed, consider any perfect complex  $\mathcal{E} \in D_{\mathrm{Qcoh}}(X)^c$  with  $\mathrm{supp}(\mathcal{E}) \subset V$ . The assumption that  $V \subset \mathcal{P}_f$  implies that  $\mathcal{E}$  has finite tor-dimension as a complex of  $f^{-1}\mathcal{O}_Y$ -modules. Indeed, it suffices to check that the complex of  $\mathcal{O}_{X,x}$ -modules  $\mathcal{E}_x$  has finite tor-dimension as an  $(f^{-1}\mathcal{O}_Y)_x \cong \mathcal{O}_{Y,f(x)}$ -module for each  $x \in X$ . For  $x \in \mathrm{supp}(\mathcal{E})$ , this follows from the fact that  $\mathcal{O}_{X,x}$  itself has finite tor-dimension as an  $\mathcal{O}_{Y,f(x)}$ -module, since  $x \in \mathrm{supp}(\mathcal{E}) \subset V \subset \mathcal{P}_f$  by assumption, while  $\mathcal{E}_x$  is zero if  $x \notin \mathrm{supp}(\mathcal{E})$ . Invoking Lemma 7.17, we conclude that  $\mathbb{R}f_*(\mathcal{E})$  is compact in  $D_{\mathrm{Qcoh}}(Y)$ . This establishes that  $V \subset Z_f$ . On the other hand, Theorem 7.15 states that the compactness locus  $Z_f$  is contained in the perfect locus  $\mathcal{P}_f$ . Since  $Z_f$  is specialization closed by definition, it follows immediately that  $Z_f$  is precisely the union of all closed subsets of  $X$  which are contained in the perfect locus  $\mathcal{P}_f$ . Indeed, if  $x \in Z_f$  then  $\overline{\{x\}} \subset Z_f \subset \mathcal{P}_f$  (by Thm. 7.15), so  $x$  is contained in a closed subset which is contained in  $\mathcal{P}_f$ .  $\square$

## 8. FURTHER EXAMPLES AND DISCUSSION

We conclude with some additional examples and a discussion of future research directions.

**8.1. Example** (Eilenberg-MacLane spectra). Consider the change-of-rings adjunction  $f^* : \mathrm{SH} \rightleftarrows \mathrm{Ho}(H\mathbb{Z}\text{-Mod}) : f_*$  associated to the map of ring spectra  $\mathbb{S} \rightarrow H\mathbb{Z}$ . Under the equivalence  $\mathrm{Ho}(H\mathbb{Z}\text{-Mod}) \cong D(\mathbb{Z})$  the right adjoint  $f_* : D(\mathbb{Z}) \rightarrow \mathrm{SH}$  sends an abelian group  $A$  to its associated Eilenberg-MacLane spectrum  $HA$ . Since  $\mathbb{Z}$  is hereditary, every object  $X \in D(\mathbb{Z})^c$  splits as a direct sum of shifts of its (finitely generated) homology groups. Since  $HA$  is not compact for any nonzero finitely generated abelian group  $A$  (by standard facts about stable cohomology operations), we conclude that if  $f_*(X) \in \mathrm{SH}^c$  then  $X = 0$  in  $D(\mathbb{Z})$ . Thus, the compactness locus of the extension-of-scalars functor  $f^* : \mathrm{SH} \rightarrow D(\mathbb{Z})$  is empty.

8.2. *Example* (Geometric fixed points). Let  $G$  be a finite group and let  $N \trianglelefteq G$  be a normal subgroup. We have discussed at length the inflation functor in equivariant stable homotopy theory (see Section 6). However, there is another prominent tensor-triangulated functor in the theory: the geometric  $N$ -fixed point functor  $\tilde{\Phi}^{N,G} : \mathrm{SH}(G) \rightarrow \mathrm{SH}(G/N)$ . It is an example of a finite localization, namely finite localization with respect to the family  $\mathcal{F}[\not\supseteq N] := \{H \leq G \mid H \not\supseteq N\}$  (cf. Example 3.9). Thinking geometrically, this is the finite localization associated to the Thomason (closed) set

$$Y := \bigcup_{H \not\supseteq N} \mathrm{supp}(G/H_+) = \{ \mathcal{P}(H, p, n) \mid 1 \leq n \leq \infty, \text{ all } p, H \not\supseteq N \} \subseteq \mathrm{Spc}(\mathrm{SH}(G)^c)$$

and we have an identification

$$\mathrm{Spc}(\mathrm{SH}(G/N)^c) \cong V := \{ \mathcal{P}(H, p, n) \mid 1 \leq n \leq \infty, \text{ all } p, H \supseteq N \} \subseteq \mathrm{Spc}(\mathrm{SH}(G)^c).$$

Applying Proposition 5.3, we can describe the compactness locus as follows:

$$Z_{\tilde{\Phi}^{N,G}} = \left( \bigcup_{(H,p): \mathcal{O}^p(H) \supseteq N} \{ \mathcal{P}(H, p, n) \mid 2 \leq n \leq \infty \} \right) \cup \left( \bigcup_{H: \mathcal{O}^p(H) \supseteq N \ \forall p} \{ \mathcal{P}(H, 1) \} \right).$$

To facilitate easy comparison with Theorem 6.2, we can rewrite this as follows: For any  $H \leq G$ , we have

- (a) If  $\mathcal{O}^p(H) \not\supseteq N$ , then  $\mathcal{P}(H, p, n) \notin Z_{\tilde{\Phi}^{N,G}}$  for all  $2 \leq n \leq \infty$ .
- (b) If  $\mathcal{O}^p(H) \supseteq N$ , then  $\mathcal{P}(H, p, n) \in Z_{\tilde{\Phi}^{N,G}}$  for all  $2 \leq n \leq \infty$ .
- (c)  $\mathcal{P}(H, 1) \in Z_{\tilde{\Phi}^{N,G}}$  if and only if  $\mathcal{O}^p(H) \supseteq N$  for all  $p$ .

In fact, we can also describe the compactness locus of the absolute geometric  $H$ -fixed point functor  $\Phi^{H,G} : \mathrm{SH}(G) \rightarrow \mathrm{SH}$  for any subgroup  $H \leq G$ . Regarding  $\Phi^{H,G} \cong \tilde{\Phi}^{H,H} \circ \mathrm{res}_H^G$  as the composite of  $f^* := \tilde{\Phi}^{H,H}$  and  $g^* := \mathrm{res}_H^G$ , it is immediate that  $\mathcal{A}_{g \circ f} \supseteq \mathcal{A}_f$  since  $g^*$  satisfies GN-duality. Moreover,  $g_* = \mathrm{coind}_H^G$  has the special property that it reflects compact objects. Indeed,  $g^*$  preserves compactness and any  $x \in \mathrm{SH}(H)$  is a direct summand of  $g^*g_*(x)$  (e.g. by [BDS15, Lem. 3.3]). Thus,  $\mathcal{A}_f = \mathcal{A}_{g \circ f}$ . Therefore, the compactness locus of  $\Phi^{H,G} : \mathrm{SH}(G) \rightarrow \mathrm{SH}$  is the subset of  $\mathrm{Spc}(\mathrm{SH}^c)$  described as follows:

- (a) If  $H$  is not  $p$ -perfect then  $\mathcal{C}_{p,n}$  is not contained in  $Z_{\Phi^{H,G}}$  for any  $2 \leq n \leq \infty$ .
- (b) If  $H$  is  $p$ -perfect then  $\mathcal{C}_{p,n}$  is contained in  $Z_{\Phi^{H,G}}$  for all  $2 \leq n \leq \infty$ .
- (c) The generic point  $\mathcal{C}_{p,1}$  ( $= \mathcal{C}_{q,1}$  for all  $q$ ) is contained in  $Z_{\Phi^{H,G}}$  if and only if  $H$  is  $p$ -perfect for all  $p$ .

8.3. *Example*. Let  $G$  be a compact Lie group. For any  $G$ -universe  $\mathcal{U}$  we can consider the stable homotopy category  $\mathrm{SH}_{\mathcal{U}}(G)$  of  $G$ -spectra indexed on  $\mathcal{U}$ . It is always compactly generated by the orbits  $\Sigma_{\mathcal{U}}^{\infty} G/H_+$  but these generators need not be rigid if the universe  $\mathcal{U}$  is not complete. Indeed, by [Lew00, Prop. 7.1],  $\Sigma_{\mathcal{U}}^{\infty} G/H_+$  is rigid in  $\mathrm{SH}_{\mathcal{U}}(G)$  if and only if  $G/H$  embeds in  $\mathcal{U}$  as a  $G$ -space. Then, defining  $\mathrm{Iso}(\mathcal{U}) := \{H \leq G \mid G/H \hookrightarrow \mathcal{U}\}$ , we can consider the subcategory

$$\mathrm{Loc}\langle \Sigma_{\mathcal{U}}^{\infty} G/H_+ \mid H \in \mathrm{Iso}(\mathcal{U}) \rangle \subset \mathrm{SH}_{\mathcal{U}}(G)$$

generated by those orbits which are rigid. It is a rigidly-compactly generated tensor-triangulated subcategory of  $\mathrm{SH}_{\mathcal{U}}(G)$  by [HPS97, p. 87]. Now, for any  $G$ -linear isometry  $i : \mathcal{V} \hookrightarrow \mathcal{U}$ , the change-of-universe functor  $i_* : \mathrm{SH}_{\mathcal{V}}(G) \rightarrow \mathrm{SH}_{\mathcal{U}}(G)$  induces a functor

$$(8.4) \quad \mathrm{Loc}\langle \Sigma_{\mathcal{V}}^{\infty} G/H_+ \mid H \in \mathrm{Iso}(\mathcal{V}) \rangle \xrightarrow{i_*} \mathrm{Loc}\langle \Sigma_{\mathcal{U}}^{\infty} G/H_+ \mid H \in \mathrm{Iso}(\mathcal{U}) \rangle$$



which is a geometric functor between rigidly-compactly generated categories. In fact, the inflation functor can be realized as a special case of this construction. Indeed, consider the inclusion  $i : \mathcal{U}^N \hookrightarrow \mathcal{U}$  of the  $N$ -fixed points in a complete  $G$ -universe  $\mathcal{U}$ . In this case, (8.4) takes the form

$$(8.5) \quad \mathrm{Loc}\langle \Sigma_{\mathcal{U}^N}^\infty G/H_+ \mid H \in \mathrm{Iso}(\mathcal{U}^N) \rangle \hookrightarrow \mathrm{SH}_{\mathcal{U}^N}(G) \xrightarrow{i_*} \mathrm{SH}(\mathcal{U}).$$

But the change-of-groups functor  $\epsilon^* : \mathrm{SH}_{\mathcal{U}^N}(G/N) \rightarrow \mathrm{SH}_{\mathcal{U}^N}(G)$  is fully faithful with essential image  $\mathrm{Loc}\langle G/H_+ \mid H \supseteq N \rangle = \mathrm{Loc}\langle G/H_+ \mid H \in \mathrm{Iso}(\mathcal{U}^N) \rangle$ . Thus, under this equivalence  $\mathrm{SH}_{\mathcal{U}^N}(G/N) \cong \mathrm{Loc}\langle G/H_+ \mid H \in \mathrm{Iso}(\mathcal{U}^N) \rangle$ , the functor (8.5) is nothing but the inflation functor  $\mathrm{infl}_{G/N}^G := i_* \circ \epsilon^*$ . The author has not pursued this but one could attempt an analysis of the adjoints and compactness locus of the more general geometric functor (8.4) associated to any  $G$ -linear isometry  $\mathcal{V} \hookrightarrow \mathcal{U}$ .

**8.6. Remark.** As discussed in Remark 3.15, we know that the relative dualizing object  $\omega_f := f^!(\mathbb{1})$  of the inflation functor  $f^* := \mathrm{infl}_{G/N}^G$  coincides with  $S^{-\mathrm{Ad}(N;G)}$  after performing the colocalization  $\Gamma^* := E\mathcal{F}(N)_+ \wedge -$ . However, it would be desirable to obtain an explicit description of  $\omega_f \in \mathrm{SH}(G)$  in general, *i.e.* before performing the colocalization. (In particular, such a description may shed light upon the relationship between the Adams isomorphism and the tom Dieck splitting theorem.) More generally, the right adjoint  $f^!$  of the categorical fixed point functor  $f_* = \lambda^N$  deserves further study. By Theorem 2.8, we know that  $\Gamma^* f^! \cong \Gamma^*(\omega_f \otimes f^*)$ , but we also know (by Theorem 1.1 and Proposition 3.2) that  $f^! \not\cong \omega_f \otimes f^*$  before colocalization, so the problem of understanding  $f^!$  does not simply reduce to understanding  $\omega_f$ .

**8.7. Remark.** In this work, we have focused on Grothendieck-Neeman duality and the construction of the Wirthmüller isomorphism — *i.e.* on the original Theorem 1.1 from [BDS16]. However, in that work there is one more theorem which completes the picture. Indeed, [BDS16, Thm. C] establishes that, for a functor  $f^*$  satisfying GN-duality and hence having the five adjoints of Theorem 1.1, the existence of one more adjoint on the left is equivalent to the existence of one more adjoint on the right, and that this is equivalent to there being an infinite sequence of adjoints in both directions. This leads to the Trichotomy Theorem [BDS16, Cor. 1.13] which states that a geometric functor between rigidly-compactly generated categories admits exactly 3, 5, or infinitely many adjoints. It would be interesting to see if one can force the next layer of adjoints, just as our Theorem 1.4 forces the first layer of adjoints after a colocalization. It would also be interesting to know if there is a dual story whereby one can force GN-duality by performing some kind of (co)localization on the source category  $\mathcal{D}$  rather than on the target category  $\mathcal{C}$ .

**8.8. Remark.** There is significant motivation for relaxing our assumption that the tensor-triangulated categories  $\mathcal{C}$  and  $\mathcal{D}$  are rigidly-compactly generated, as this would potentially enable the theory developed here (and in [BDS16]) to embrace several important examples in chromatic and motivic homotopy theory. For example, although the stable  $\mathbb{A}^1$ -homotopy category  $\mathrm{SH}^{\mathbb{A}^1}(S)$  is compactly generated, it is unlikely to be generated by rigid objects if the base scheme  $S$  is positive dimensional (cf. Example 2.2(e) and Example 8.3). On the other hand, the  $K(n)$ -local stable homotopy category (see [HS99]) is a prominent example of a tensor-triangulated category which is *almost* rigidly-compactly generated: it is generated by a set of compact-rigid objects and the compact objects are rigid, but rigid objects need not



be compact (e.g. the unit object is not compact). Generalizing our treatment of the Wirthmüller isomorphism (and the duality results of [BDS16]) to cover such examples of non-unital algebraic stable homotopy categories may lead to connections with Gross-Hopkins duality (à la [DGI11]) and Hopkins and Lurie’s work on “ambidexterity” in  $K(n)$ -local stable homotopy theory (see [HL13, Sec. 4–5]). This will be pursued in future work. Since the category of colocal objects appearing in Theorem 2.8 is precisely such an example of a non-unital algebraic stable homotopy category, the author has some optimism that there may be fruit down this road.

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